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Caiqin SONG

Solutions to Linear Matrix Equations and Their Applications

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About the Author

Caiqin Song, now is an associate professor and Master's supervisor. She received her PhD from East China Normal University in June 2012. The main research direction is the explicit method of linear matrix equations, the iterative algorithm of linear matrix equations, the explicit method of quaternion matrix equations, and the problem of the semi-tensor product of matrix, etc. She has published more than 30 academic papers in international academic journals. She once presided over one National Natural Science Foundation of China (No. 11501246), one first-class grant and one second-class grant from the China Postdoctoral Science Foundation (No. 2017M610243, 2013M541900), and one Shandong Provincial Natural Science Foundation (No. ZR2020MA052).

Preface

Solving linear matrix equations has experienced decades of development and has become a relatively mature scientific field. The study of linear matrix equations needs to focus on two important elements: one is the sufficient condition or necessary and sufficient condition of the linear matrix equation having a solution; the other is the explicit solution expression or iterative algorithm formula. The topic on linear matrix equations includes the solvable conditions of linear matrix equations, mainly including sufficient conditions and necessary and sufficient conditions; explicit method of linear matrix equations; iterative algorithms of linear matrix equations and comparison with various types of existing algorithms in time and accuracy; the application of linear matrix equation in the field of control. The knowledge of linear matrix equations is becoming more and more necessary. It has been applied in many engineering branches, such as robust control, observer design, and attitude control of spacecraft.

This book is written for a wider audience and has three main purposes. Firstly, for researchers who want to consider the linear matrix equation, but lack of the knowledge about linear matrix equations. In this case, this book can be used as a primer book. Secondly, this book can provide its control model for researchers who have many important and good research results in the field of the linear matrix equation. Thirdly, this book provides textbooks or reference books for graduate students in related fields. In order to achieve the above purpose, this book provides two types of iterative algorithms for some linear matrix equations, which are used to find the numerical solutions and least squares solutions of several types of linear matrix equations, and provides an explicit method for several types of linear matrix equations.

This book has several characteristics: Firstly, the book contains a number of recent research results, giving a finite iterative algorithm for coupled operator matrix equations with sub-matrix constraints and a least squares iterative algorithm for coupled operator matrix equations. Secondly, the control model and the application of the algorithm to the linear matrix equation are given. The iterative

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algorithms and explicit solutions provided in this book are very useful for further research on control problems. This book is not only easy to understand, but also provides a large number of cybernetic models and numerical examples to illustrate the superiority and practicality of the algorithm.

Caiqin SONG
July 29, 2021

Notations

$\mathfrak{C}$: the set of all complex numbers
$\Re$: the set of all real numbers
$\Re^{m\times n}$: the set of all the real matrix $m\times n$
$\mathfrak{C}^{m\times n}$: the set of all the complex matrix $m\times n$
$\bar{A}$: the conjugate of matrix A
A^T: the transpose of matrix A
A^*: the conjugate transpose of matrix A
A^{-1}: the inverse of matrix A
I_n: the identity matrix with $n\times n$
$0_{m\times n}$: the zero matrix with $m\times n$
$\det(A)$: the determinant of matrix A
$r(A)$: the rank of matrix A
$tr(A)$: the trace of matrix A
$\rho(A)$: the spectral radius of matrix A
$\lambda(A)$: the set of character value on matrix A
$\|x\|_2$: Euclid length of vector x
$\|A\|_2$: the spectral norm of matrix A
$\|A\|_F$: the Frobenius norm of matrix A
$A\geq B$: represents A and B with same size satisfying $a_{ij}\geq b_{ij}$
$\cup$: Collective union
$\cap$: Collection of crosses
$\emptyset$: empty set
$\Longleftrightarrow$: equivalence
a_i: the ith column of A
$\mathrm{vec}\,(A)=(a_1^T, a_2^T, \ldots, a_n^T)^T$: the vec operator of matrix A
$\langle A, B\rangle$: the inner product of matrices A and B
$LC^{m\times n, p\times q}$: the set of linear operators from $C^{m\times n}$ onto $C^{p\times q}$
$\mathcal{I}$: the identity operator on $C^{m\times n}$
$I[1, N]$: the integer set from 1 to N

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Chapter 1

Introduction

In this book, we will derive three parts to consider the solutions to linear equations. They are finite iterative algorithms to the linear matrix equations, MCGLS iterative algorithms to the linear matrix equations, and explicit solutions to the linear matrix equations, respectively.

1.1 The First Part: Finite Iterative Algorithm to Linear Matrix Equation

In chapter 2, we consider a finite iterative algorithm to the coupled Sylvester-transpose matrix equations. It is well-known that Sylvester and Lyapunov matrix equations are very important since they play a fundamental role in the various field of engineering theory, particularly in control systems. The numerical solutions of Sylvester and Lyapunov matrix equations have been addressed in a large number of literature. Zhou and Duan[1–3] established the solutions of several generalized Sylvester matrix equations. Zhou *et al.*[4] proposed a gradient-based iterative algorithm for solving the general coupled Sylvester matrix equations with weighted least squares solutions. In[5], the general parametric solutions to a family of generalized Sylvester matrix equations arising in linear system theory was presented by using the so-called generalized Sylvester mapping which had some elegant properties. With the help of the Kronecker map, Wu *et al.*[6] introduced a complete, general, and explicit solution to the Yakubovich matrix equation $V - AVF = BW$, with F in an arbitrary form. Also using the Kroneker map, an explicit solution for the matrix equation $XF - AX = C$ was established[7].

Moreover, Ding and Chen presented the hierarchical gradient iterative (HGI) algorithms for general matrix equations[10,11] and hierarchical least squares iterative (HLSI) algorithms for generalized coupled Sylvester matrix equations and general coupled matrix equations[12,13]. The HGI algorithms[10,11] and HLSI algorithms[10,13,14] for solving general (coupled) matrix equations are innovational and computationally efficient numerical ones and are proposed based on the hierarchical identification principle[12,15] which regards the unknown matrix as the system

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parameter matrix to be identified. In[16], Dehghan and Hajarian proposed an iterative algorithm for solving the second-order Sylvester matrix equation

$$EVF^2 - AVF - CV = BW. \tag{1.1.1}$$

Recently Dehghan and Hajarian[17–20] introduced several iterative methods to solve (coupled) matrix equations such as (coupled) Sylvester matrix equations over reflexive, anti-reflexive, and generalized bisymmetric matrices.

In chapter 3, we consider the finite iterative algorithm to the coupled operator matrix equations with sub-matrix constrained. Chen[59] puts forward the applications of the generalized centrosymmetric matrix in three major areas: the altitude estimation of a level network, an electric network, and the structural analysis of trusses. Coupled matrix equations have numerous applications in stability theory, control theory, perturbation analysis, signal processing, and image restoration. For example, in stability analysis of linear jump systems with Markovian transitions, the continuous-time coupled Lyapunov matrix equations

$$A_i^T P_i + P_i A_i + Q_i + \sum_{j=1}^{N} p_{ij} P_j = 0, \quad Q_i > 0, i \in I[1, N], \tag{1.1.2}$$

and

$$P_i = A_i^T \Big(\sum_{j=1}^{N} p_{ij} P_j \Big) A_i + Q_i, \quad Q_i > 0, i \in I[1, N], \tag{1.1.3}$$

are often encountered[58,65,138], where $P_i > 0, i \in I[1, N]$ are the matrices to be determined. The coupled Sylvester matrix equations arise in computing error bounds for eigenvalue and eigenvalue space of the generalized eigenvalue problem[33]

$$S - \lambda T = \begin{pmatrix} A & -C \\ 0 & B \end{pmatrix} - \lambda \begin{pmatrix} D & -F \\ 0 & E \end{pmatrix}, \tag{1.1.4}$$

in computing deflating subspace of the same problem, and in computing certain decomposition of the transferable matrix arising in control theory.

Many papers have studied several matrix equations[8,18,35,36,38,40,43–46,51,52,56,66,76,100,126]. But with the development of economy, science, and technology, many researchers pay close attention to transpose matrix equations and conjugate matrix equations. Sylvester transpose matrix equation can be used to solve many control problems such as pole assignment[60], eigenstructure assignment[61,166–168], and robust fault detection[62]. Recently, in[64], minimal residual method was reported for solving CT Sylvester matrix equations and DT Sylvester matrix equations. In[16,63,67,169], finite iterative algorithm was constructed for solving the second-order Sylvester matrix equation $EVF^2 - AVF - CV = BW$. In[47,48], Wang investigated the centrosymmetric solution to the matrix equation systems $A_1X = C_1, A_3XB_3 = C_3$. Moreover, some necessary and sufficient conditions, parameterize general solution and the maximal and minimal ranks of the general solution to the mixed Sylvester matrix equations and coupled Sylvester matrix equations are investigated[68–71].

Generalized Sylvester matrix equation, quaternion matrix equation, and operator matrix equation have been studied[72–75].

By applying the so-called generalized Kroneker map which had some elegant properties, Wu *et al.*[77] discussed the closed-form solution to the generalized Sylvester-conjugate matrix equation. Zhou and Duan[80] proposed a gradient-based iterative algorithm for solving general coupled Sylvester matrix equations arising in linear systems theory. By using the so-called generalized Sylverter mapping, right coprime factorization, and Bezout identity associated with the certain polynomial matrix, Zhou *et al.*[81] investigated the problem of parameterizing all solutions to the polynomial Diophantine matrix equation and the generalized Sylverter matrix equation. It is shown that the provided solutions can be parameterized as soon as two pairs of polynomial matrices satisfying the right coprime factorization and Bezout identity are obtained. Based on the conjugate gradient searching principle and its dual form, Li *et al.*[21] presented two algorithms for solving the minimal norm least squares solution to a general linear equation $\sum_{i=1}^{r} A_i X B_i + \sum_{j=1}^{s} C_j X^T D_j = E$. Between these two methods, the first one is to minimize the spectral radius of the iteration matrix, and explicit expression for the optimal step size is obtained. The second one is to minimize the square sum of the F-norm of the error matrices produced by the algorithm. The solvability, existence of unique solution, closed-form solution and numerical solution of matrix equation $X = Af(X)B + C$ with $f(X) = X^T, f(X) = \overline{\mathrm{X}}$ and $f(X) = X^H$, where X was the unknown matrix, were investigated[80]. In[22], the following matrix equation

$$\sum_{i=1}^{r} A_i X B_i + \sum_{j=1}^{s} C_j X^T D_j = E, \tag{1.1.5}$$

was considered, where $A_i, B_i, C_j, D_j, i = 1, \ldots, r; j = 1, \ldots, s,$ and E were some known constant matrices of appropriate dimensions and X was a matrix to be determined. In[67], the solution of the generalized Sylvester-transpose matrix equation $AXB + CX^T D = E$ was presented by the iterative algorithm. The Moore–Penrose generalized inverse was used in[24] to study the matrix equation $AX + X^T B = C$ and the explicit solutions to this matrix equation were given.

1.2 The Second Part: MCGLS Iterative Algorithm to Linear Matrix Equation

In chapter 4, we present MCGLS iterative algorithm to the linear conjugate matrix equation. Many problems in mathematics, physics, and engineering lead to solving linear matrix equations. For instance, a digital filter can be characterized by state variable equations

$$x(n+1) = Ax(n) + bu(n), \tag{1.2.1}$$

and

$$y(n) = cx(n) + du(n). \tag{1.2.2}$$

The solutions K and W of the Lyapunov matrix equations

$$K = AKA^T + bb^T, \tag{1.2.3}$$

and

$$W = A^T WA + c^T c, \tag{1.2.4}$$

can analyzed the quantization noise generated by a digital filter[83]. The Lyapunov and (generalized) Sylvester matrix equations appear in robust control[84], neural network[85] and singular system control[86]. It can be shown that certain control problem, such as pole/eigenstructure assignment and observe design[5,87] of (1.2.1), are closely related with the following generalized Sylvester matrix equation

$$\sum_{k=1}^{p} A_k X B_k + \sum_{j=1}^{q} C_j Y D_j = E. \tag{1.2.5}$$

So far many techniques have been implemented for obtaining solutions of various linear matrix equations[23,37,39,41,42,44,48–50,68,69]. In[37], the least squares solution of the generalized Sylvester matrix equation

$$AXB + CYD = E, \tag{1.2.6}$$

was studied for symmetric arrowhead matrices with the least norm. By applying the hierarchical identification principle, Ding and Chen introduced gradient based iterative algorithms to solve (coupled) generalized Sylvester matrix equations, nonlinear systems[11,13,14,91,92]. Recently, in[89,90,93,94], some efficient extended conjugate gradient methods were proposed for solving the least squares solution of several linear matrix equations. Very recently, the matrix forms of CGS, BiCOR, CORS, Bi-CGSTAB and finite iterative methods were investigated for solving various linear matrix equations[78,95–98,101–104]. Based on the idea of the paper[37,41,42,77,78,92–98,101–104], this chapter focuses on the goal of MCGLS algorithm for computing the least square solution of linear conjugate matrix equation.

In chapter 5, based on the theory of chapter 5, we present MCGLS iterative algorithm for solving the generalized Sylvester conjugate transpose matrix equation. The generalized Sylvester conjugate transpose matrix equation refers to linear systems whose state evolution depends on both the state and its conjugate. Several reasons can be found to study this class of linear systems, for instance, they are naturally encountered in linear dynamical quantum systems theory and any real-valued linear systems with lower dimensions. Therefore, this kind of matrix equation has many applications in some fundamental problems such as state response, controllability, observability, stability, pole assignment, linear quadratic regulation, and state observer design[3,84,105–112]. So far many techniques have been implemented for obtaining solutions to various linear matrix equations[113–119].

Recently, in[11,29,90,93], some efficient extended conjugate gradient methods were proposed for solving the least squares solution of several linear matrix equations. Based on the idea of the paper[11,48–50,68,69,77,78,89,90,93–98], this chapter is concerned with the least square solution of the generalized Sylvester conjugate transpose matrix equation.

In chapter 6, based on chapters 4 and 5, we consider the least squares solutions to coupled linear operator systems by using MCGLS iterative algorithm. The least squares problems have wide applications in particle physics and geology, digital image and signal processing, inverse Sturm–Liouville problem, inverse problems of vibration theory, and control theory[105–109].

As we all know, matrix equations often arise from control theory, system theory, and stability analysis[83,117–119]. Many authors have received much attention on how to solve these matrix equations. Therefore, some different methods for solving matrix equations were established. The hierarchical identification principle was presented by Ding and Chen[10,11,13,14]. In order to solve coupled matrix equations, the gradient-based iterative algorithms were established by combining the hierarchical identification principle and the gradient iterative method of the simple matrix equations[22,120,121]. For more references, one can refer to[4,44,48–50,67–69,88,115–119,122–124]. To improve the previous result, Zhou developed the weighted least squares iterative for solving the coupled matrix equation in[4].

Another important method for solving the least squares problem of matrix equations is the CGLS method. Originating from the conjugate gradient method, this method has become more popular. The reason of presented CGLS iterative method is that the residual and the relative error of our iterative algorithm are smaller than those of LSQR-M iterative algorithm[90,98,125].

1.3 The Third Part: Explicit Solutions to Linear Matrix Equation

In chapter 7, we consider the explicit solution to the nonhomogeneous Yakubovich matrix equation. This nonhomogeneous Yakubovich matrix equation plays an important role in output regulator design for time-invariant linear systems[141] and eigenstructure assignment for second-order linear systems[130,131]. Obviously, the well-known Yakubovich matrix equation

$$X - AXB = CY, \tag{1.3.1}$$

and generalized Sylvester matrix equation

$$AX - EXF = CY, \tag{1.3.2}$$

are the special cases of the nonhomogeneous Yakubovich matrix equation, respectively. If the coefficient matrices satisfy some relations, the generalized Sylvester matrix equation can be transformed into the Yakubovich matrix equation and the solutions to these two matrix equations are equivalent. As we have known, the generalized discrete Sylvester matrix equation is closely related to many problems in conventional linear control system theory, such as pole/eigenstructure assignment design, Luenberger-type observer design, robust fault detection[132,136,137], and so on. Finding the complete parametric solutions to the matrix equation is of extreme importance because many problems are referred to as design freedom, such as robustness in control system design.

Due to the above-mentioned applications, the generalized Sylvester matrix equations (1.3.1) and (1.3.2) have been studied by many authors. When the coefficient matrix F is in Jordan form, Duan[133] has proposed a complete parametric solution to the generalized Sylvester matrix equation (1.3.2). However, this solution is not in a direct, explicit form but in a recursive form. Meanwhile, when the matrix F is also in Jordan form and the matrix triple (E, A, B) is assumed to be R-controllable, Duan[53] has proposed a complete and explicit solution by applying the right coprime factorization of the input-state transfer function $(sE - A)^{-1}B$. For the generalized discrete Sylvester matrix equation (1.3.1), an explicit, analytical, and complete solution is obtained in[6,47]. In[147], the proposed solution is in a very neat form and can be immediately obtained as soon as a series of matrices $D_i, i = 0, 1, \ldots, \omega$. In[6], the explicit solution can be obtained by using the Kronecker map. The dual form of the matrix equation (1.3.1), which is called the generalized Sylvester matrix equation, has also been well studied[134,144,149]. For instance, in[144], an analytical and restriction-free solution is established when the coefficient matrix F is in a Jordan form. In order to obtain this solution, it needs to computer an inversion, carries out an orthogonal transformation, and solve a series of linear equation groups. When the matrix F is also in Jordan form, two solutions of the matrix equation in[134] are proposed. One solution is in an iterative form, the other solution is in an explicit parametric form. In order to obtain the second explicit parametric solution, one needs to carry out a right coprime factorization of the polynomial matrix $(sI - A)^{-1}B$ in the case of the eigenvalues of the Jordan matrix F being undetermined or a series of singular value decomposition when the eigenvalues of F are known. Moreover, some necessary and sufficient conditions, parameterize the general solution and the maximal and minimal ranks of the general solution to the mixed Sylvester matrix equations and coupled Sylvester matrix equations are investigated[68–71]. The matrix equations which are studied in the current paper have included many linear matrix equations as special cases, such as discrete Sylvester matrix equation, generalized Sylvester matrix equation, discrete Lyapunov matrix equation, and so on[6,134,144]. The idea of the present chapter comes from the references[6,53,133,134,139,142,145–150].

In chapter 8, we consider the explicit solutions to the nonhomogeneous Yakubovich-transpose matrix equation. The matrix equations $AX - XB = C$ and $X - AXB = C$ play important role in stability theory and control theory[132,140,143,151,152,165,170,171]. When $A = B^T$ (the transpose of B), the matrix equations are the well-known Lyapunov matrix equation and Stein matrix equation. Many control problems, for example, pole assignment[153–155] and eigenstructure assignment[54], are closely related to the second-order nonhomogeneous generalized Sylvester matrix equation $MXF^2 + DXF + KX = BY + R$, where (X, Y) is a pair of matrices to be determined and other matrices are the known matrices. When $R=0$, it becomes the second-order homogeneous matrix equation $MXF^2+DXF+KX=BY$. Other matrix equations such as the coupled Sylvester matrix equations and the Riccati equation have also been found numerous applications in control theory. For more related introduction, see[72,156–161] and the references therein. Furthermore, another kind of transpose matrix equation, for example, $AX+X^TB=C$, $AX\pm X^TA^T=B$ and $AXB+CX^TD=E$, have received much attention in the literature over the past several decades[21,22,24,120,162–164]. In these references, the necessary and sufficient conditions for the existence of solutions, the solvability and closed-form solutions of the above transpose matrix equations are obtained by using SVD, GSVD, Moore–Penrose generalized inverse method and iterative algorithm.

In chapter 9, we consider the explicit solution to the generalized Sylvester matrix equation. In the field of linear algebra, there is another type of similarity which is called consimilarity. For two complex square matrices A and B, there exists a nonsingular complex matrix P such that $A=P^{-1}B\overline{P}$[171]. Then A and B is called consimilarity. Consimilarity of complex matrices is referred to studying an antilinear operator for different bases in complex vector space. The consimilarity theory of complex matrices plays an important role in the research of modern quantum theory[173]. For this reason, the complex matrix equations $X-AXB=CY$ and $AX-XB=CY$ have been respectively extended to complex conjugate matrix equation $X-A\overline{X}B=CY$ and $AX-\overline{X}B=CY$ by applying consimilarity. In the quaternion field, the consimilarity of quaternion matrices is also defined in[174]. Similar to the linear matrix equation $AX-XB=CY$, the quaternion matrix equation $XB-AX=CY$ has also been generalized to the quaternion $\jmath$-conjugate matrix equation $XB-A\widehat{X}=CY$ by applying consimilarity over the quaternion field. Traditionally, quaternion algebra[175,176] has been extensively used in computer graphics[177] and aerospace problems[178], due to its compact notation, moderate computational requirements, and avoidance of singularities associated to 3×3 rotation matrices[179]. So many researchers pay close attention to the quaternion matrix equation problem. For example, Ning and Wang[180] have studied the maximal and minimal ranks of a 2×2 partitioned quaternion matrix. Huang[181] has established some necessary and sufficient conditions for the existence of a unique solution to the quaternion matrix equation $\sum_{i=0}^{k}A^iXB_i=E$. By applying the complex representation of the quaternion matrix, the Moore–Penrose generalized inverse, and the Kronecker product of matrices, the least squares solution of the quaternion $\jmath$-conjugate matrix equation $X-A\widehat{X}B=C$ (where $\widehat{X}$ denotes the $\jmath$-conjugate of quaternion matrix X) with the least norm[182] has been discussed. Some necessary and sufficient conditions for the existence of the quaternion matrix equations $AXB+CYD=E$, $(AX,XC)=(B,D)$, $AXB=C$, have been proposed and the solutions have been derived in the references[48,49,183]. Song *et al.*[160,184] have studied the explicit solutions to the quaternion $\jmath$-conjugate matrix equation $X-A\widehat{X}B=C$ and $XF-A\widehat{X}=CY$, where the coefficient matrix A is a block diagonal form over a complex field.

In recent years, some other linear matrix equations and nonlinear matrix equations have also been attached. By using the so-called generalized Sylvester mapping, right coprime factorization, and Bezout identity associated with the certain polynomial matrix, Zhou *et al.*[81] have investigated the problem of parameterizing all solutions to the polynomial Diophantine matrix equation and the generalized Sylverter matrix equation. It is shown that the provided solutions can be parameterized as soon as two pairs of polynomial matrices satisfy the right coprime factorization. The solvability, existence of unique solution, closed-form solution, and numerical solution of matrix equation $X = Af(X)B + C$ with $f(X) = X^T, f(X) = \overline{X}$ and $f(X) = X^H$, where X is the unknown matrix, are investigated[80]. A complete, general, and explicit solution[81] to the generalized Sylvester matrix equation $AX - XF = BY$, with the coefficient matrix F in a companion form, is proposed. By introducing and studying a matrix operator on complex matrices, some sufficient conditions and necessary conditions for the existence of positive definite solutions of the matrix equation $X + A^H\overline{X}^{-1}A = I$[185] are also proposed. In addition, Wu *et al.*[6,7] have constructed some explicit expressions of the solution to the matrix equation $AX - XB = C$ and $X - AXB = CY$ by applying the Kronecker map. It is shown that there exists a unique solution if and only if $A\overline{A}$ and $B\overline{B}$ have no common eigenvalues.

In chapter 10, we consider the explicit solutions to the equations

$$AX + X^TB = C, \tag{1.3.3}$$

and

$$AX + X^TB = CY, \tag{1.3.4}$$

where $A, B, C \in \mathcal{R}^{n\times n}$ are given $n \times n$ real matrices and $X, Y \in \mathcal{R}^{n\times n}$ are unknown $n \times n$ real matrices to be determined. There are named the Sylvester transpose matrix and the generalized Sylvester transpose matrix equation, respectively. If $X = X^T$, the equations (1.3.3) and (1.3.4) reduce to the well-known Sylvester (or generalized Sylvester) matrix equation, which has many important applications in control theory. Therefore, many researchers focus on the solutions to these two equations, for details, please see[30–32]. Because (1.3.3) and (1.3.4) are obtained by changing the second unknown matrix of the (generalized) Sylvester matrix equation, they are called the (generalized) Sylvester transpose matrix equation.

The traditional method for solving the equation (1.3.3) is to transformed it into its equivalent form $\mathcal{A}x = c$, where $\mathcal{A} = I_n \otimes A + (B^T \otimes I_n)P_{nn}$, $c = \mathrm{vec}(C)$ and $x = \mathrm{vec}(X)$. Therefore, the unique solution to equation (1.3.3) can be obtained as $x = \mathcal{A}^{-1}c$[55]. Recently, a lot of methods have been constructed for calculating explicit solutions to (1.3.3) and are briefly classified as follows.

The method of applying the matrix decomposition was given in[189]. The method by using GSVD, CCD, and projective theorem was presented to discuss the least square solution to the transpose equation $A^TXB + B^TX^TA = D$ [162]. The method of using a generalized inverse was proposed to investigate the explicit solution to the equation (1.3.3)[163]. Also, some solvability was provided for the equations

$A^\star X + X^\star A = B$ and $A^\star X + X^\star A = 0$ and the expressions of the solutions were given to the two matrix equations above[159,190]. Some necessary and sufficient conditions were presented for the existing unique solution to the generalized $\star$-Sylvester matrix equation and they were in terms of the Kronecker canonical form of the matrix pencil $A + \lambda B$[191]. The solvability of the Sylvester matrix equation was provided by Desouza and Bhattacharyya[31] and it demonstrated that the solution existed if and only if A and B were similar. For a more related introduction, see[30,157,163,190–194] and the references therein. In addition, many works were devoted to calculating the numerical solutions to the Sylvester transpose equation. Here, we are not concerned with the numerical solutions to the linear transpose matrix equations and we suggested the readers to see[23,120] and their references therein. Besides the topics above on linear matrix equations, some other topics were also focused on. For instance, the solutions to the (generalized) Sylvester matrix equation over the real field, complex field, and quaternion field, for details, please see[1,21,22,50,145,149,195–201].

However, the explicit solutions to the matrix equations (1.3.3) and (1.3.4) have not yet been given in the literature. The difficulty of solving equations (1.3.3) and (1.3.4) is that there are unknown matrices X and X^T. It is clear that the form of an unknown matrix is not united. Inspired by[30,50,139,186,198,199], in this chapter, we'll adopt the Kronecker map and Sylvester sum to study explicit solutions to the equations (1.3.3) and (1.3.4). We'll propose the solvability and necessary and sufficient conditions for these two equations. Moreover, we'll offer three forms of solutions to equations (1.3.3) and (1.3.4) and one of the solutions is Jameson's Theorem. We also establish the corresponding algorithms for the matrix equations above and two algorithms are proposed for solving the equation (1.3.3) when A or B is nonsingular. As the application of our presented algorithm, we give the design of the time-varying linear system in control theory.

In short, this book wants to build a working platform for readers, provide a benchmark, and become a cornerstone for further learning, application, and development of matrix equations. The author is very talented and shallow, and omissions are inevitable. Readers and related experts are welcome to give us advice.

Chapter 2

Finite Iterative Algorithm to Coupled Transpose Matrix Equations

In this chapter, we consider the following two problems of the coupled Sylvester-transpose matrix equations.

Problem 2.1. Given matrices $A_{i\eta} \in \Re^{m_i \times l_\eta}$, $B_{i\eta} \in \Re^{n_\eta \times p_i}$, $C_{i\eta} \in \Re^{m_i \times n_\eta}$, $D_{i\eta} \in \Re^{l_\eta \times p_i}$, and $F_i \in \Re^{m_i \times p_i}$, $i \in I[1,N], \eta \in I[1,p]$, find the matrix group $(X_1, X_2, \ldots, X_p)$ with $X_\eta \in \Re^{l_\eta \times n_\eta}, \eta \in I[1,p]$ such that

$$\sum_{\eta=1}^{p} (A_{i\eta} X_\eta B_{i\eta} + C_{i\eta} X_\eta^T D_{i\eta}) = E_i, \quad i = 1, 2, \ldots, p. \tag{2.0.1}$$

Problem 2.2. Let problem 2.1 be consistent, and its solution group set is denoted by S_r. For a given matrix group $(\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_p)$ with $\widehat{X}_\eta \in R^{l_\eta \times n_\eta}, \eta \in I[1,p]$, find $(\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_p) \in S_r$ with $\widetilde{X}_\eta \in \Re^{l_\eta \times n_\eta}, \eta \in I[1,p]$ such that

$$\sum_{j=1}^{p} \|\widetilde{X}_j - \widehat{X}_j\|^2 = \min_{(X_1, \ldots, X_p) \in S_r} \left\{ \sum_{j=1}^{p} \|X_j - \widehat{X}_j\|^2 \right\}. \tag{2.0.2}$$

In[21,22], the following linear equation

$$\sum_{i=1}^{r} A_i X B_i + \sum_{j=1}^{s} C_j X^T D_j = E, \tag{2.0.3}$$

where $A_i, B_i, C_j, D_j, i = 1, \ldots, r; j = 1, \ldots, s.$ and E were some known constant matrices of appropriate dimensions and X was a matrix to be determined, was considered. In[23], the special case of (2.0.3), which was matrix equation $AXB + CX^T D = E$, was considered by an iterative algorithm. A more special case of (2.0.3), namely, the matrix equation $AX + X^T B = C$, was investigated by Piao *et al.*[24]. The Moore–Penrose generalized inverse was used in[24] to find explicit

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solutions to this matrix equation. Compared with the above references, in this chapter, we investigate the finite iterative algorithm to the matrix equation (2.1.1).

2.1 Finite Iterative Algorithm and Convergence Analysis

In this section, we consider the following coupled Sylvester-transpose matrix equations

$$\sum_{\eta=1}^{p}(A_{i\eta}X_{\eta}B_{i\eta}+C_{i\eta}X_{\eta}^{T}D_{i\eta})=F_{i}, i\in I[1,N], \tag{2.1.1}$$

where $A_{i\eta}\in\Re^{m_i\times l_\eta}, B_{i\eta}\in\Re^{n_\eta\times p_i}$, $C_{i\eta}\in\Re^{m_i\times n_\eta}, D_{i\eta}\in\Re^{l_\eta\times p_i}$, $F_i\in\Re^{m_i\times p_i}, i\in I[1,N]$, $\eta\in I[1,p]$, are the given known matrices, and $X_\eta\in\Re^{l_\eta\times n_\eta}, \eta\in I[1,p]$ are the matrices to be determined.

When the matrix equations are consistent, we propose the following iterative algorithm to solve it.

Algorithm 2.1

Step 1. Input matrices $A_{i\eta}\in\Re^{m_i\times l_\eta}, B_{i\eta}\in\Re^{n_\eta\times p_i}, C_{i\eta}\in\Re^{m_i\times n_\eta}$, $D_{i\eta}\in\Re^{l_\eta\times p_i}$, $F_i\in\Re^{m_i\times n_i}, i\in I[1,N]$, and arbitrary matrices $X_\eta(1)\in\Re^{l_\eta\times n_\eta}$, $\eta\in I[1,p]$; Calculate

$$R_i(1)=F_i-\sum_{\eta=1}^{p}[A_{i\eta}X_\eta(1)B_{i\eta}+C_{i\eta}X_\eta^T(1)D_{i\eta}], i\in I[1,N],$$

$$P_\eta(1)=\sum_{i=1}^{N}[A_{i\eta}^TR_i(1)B_{i\eta}^T+D_{i\eta}R_i^T(1)C_{i\eta}], \eta\in I[1,p].$$

Step 2. If $R_i(k)=0, i\in I[1,N]$ then stop; else, $k=k+1$.
Step 3. Calculate

$$X_\eta(k+1)=X_\eta(k)+\frac{\sum_{\omega=1}^{N}\|R_\omega(k)\|^2}{\sum_{\tau=1}^{p}\|P_\tau(k)\|^2}P_\eta(k), \eta\in I[1,p],$$

$$\begin{aligned}R_i(k+1)&=F_i-\sum_{\eta=1}^{p}[A_{i\eta}X_\eta(k+1)B_{i\eta}+C_{i\eta}X_\eta^T(k+1)D_{i\eta}]\\&=R_i(k)-\frac{\sum_{\omega=1}^{N}\|R_\omega(k)\|^2}{\sum_{\tau=1}^{p}\|P_\tau(k)\|^2}\sum_{\eta=1}^{p}[A_{i\eta}P_\eta(k)B_{i\eta}\\&\quad+C_{i\eta}P_\eta^T(k)D_{i\eta}], i\in I[1,N],\end{aligned}$$

$$P_\eta(k+1)=\sum_{i=1}^{N}[A_{i\eta}^TR_i(k+1)B_{i\eta}^T+D_{i\eta}R_i^T(k+1)C_{i\eta}] \\ +\frac{\sum_{\omega=1}^N\|R_\omega(k+1)\|^2}{\sum_{\omega=1}^N\|R_\omega(k)\|^2}P_\eta(k),\eta\in I[1,p].$$

Step 4. Go to step 2.

Now we introduce some properties of the algorithm above.

Lemma 2.1. Suppose that the sequences $P_\eta(k),\eta\in I[1,\ p]$ and $R_i(k),\ i\in I[1,N]$ are generated by algorithm 2.1, then the following relation holds

$$\begin{aligned}&\sum_{i=1}^{N}tr[R_i^T(k+1)R_i(j)]\\ &=\sum_{i=1}^{N}tr[R_i^T(k)R_i(j)]-\frac{\sum_{\omega=1}^N\|R_\omega(k)\|^2}{\sum_{\tau=1}^p\|P_\tau(k)\|^2}\sum_{\eta=1}^{p}tr[P_\eta^T(k)P_\eta(j)]\\ &+\frac{\sum_{\omega=1}^N\|R_\omega(k)\|^2\sum_{\omega=1}^N\|R_\omega(j)\|^2}{\sum_{\tau=1}^p\|P_\tau(k)\|^2\sum_{\omega=1}^N\|R_\omega(j-1)\|^2}\sum_{\eta=1}^{p}tr[P_\eta^T(k)P_\eta(j-1)].\end{aligned}\tag{2.1.2}$$

Proof. Because all matrices in algorithm 2.1 are real, we can write

$$\begin{aligned}&\sum_{i=1}^{N}tr[R_i^T(k+1)R_i(j)]\\ &=\sum_{i=1}^{N}tr\Big[(R_i(k)-\frac{\sum_{\omega=1}^N\|R_\omega(k)\|^2}{\sum_{\tau=1}^p\|P_\tau(k)\|^2}\sum_{\eta=1}^{p}(A_{i\eta}P_\eta(k)B_{i\eta}\\ &+C_{i\eta}P_\eta^T(k)D_{i\eta}))^TR_i(j)\Big]\\ &=\sum_{i=1}^{N}tr[R_i^T(k)R_i(j)]-\frac{\sum_{\omega=1}^N\|R_\omega(k)\|^2}{\sum_{\tau=1}^p\|P_\tau(k)\|^2}\\ &\times\sum_{i=1}^{N}tr\left[\sum_{\eta=1}^{p}(B_{i\eta}^TP_\eta^T(k)A_{i\eta}^T+D_{i\eta}^TP_\eta(k)C_{i\eta}^T)R_i(j)\right].\end{aligned}$$

In addition, by applying some properties of the trace of a matrix one has

$$\begin{aligned}&\sum_{i=1}^{N}tr\left[\sum_{\eta=1}^{p}\left(B_{i\eta}^TP_\eta^T(k)A_{i\eta}^T+D_{i\eta}^TP_\eta(k)C_{i\eta}^T\right)R_i(j)\right]\\ &=\sum_{i=1}^{N}tr\left[\sum_{\eta=1}^{p}\left(P_\eta^T(k)A_{i\eta}^TR_i(j)B_{i\eta}^T+P_\eta(k)C_{i\eta}^TR_i(j)D_{i\eta}^T\right)\right].\end{aligned}$$

By changing order of this double sum, one has from algorithm 2.1,

$$\begin{aligned}
&\sum_{i=1}^{N} tr\Big[\sum_{\eta=1}^{p}\left(B_{i\eta}^{T}P_{\eta}^{T}(k)A_{i\eta}^{T}+D_{i\eta}^{T}P_{\eta}(k)C_{i\eta}^{T}\right)R_i(j)\Big]\\
&\quad=\sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N}\left(P_{\eta}^{T}(k)A_{i\eta}^{T}R_i(j)B_{i\eta}^{T}+P_{\eta}(k)C_{i\eta}^{T}R_i(j)D_{i\eta}^{T}\right)\Big]\\
&\quad=\sum_{\eta=1}^{p} tr\Big[P_{\eta}^{T}(k)\Big(\sum_{i=1}^{N}A_{i\eta}^{T}R_i(j)B_{i\eta}^{T}\Big)+P_{\eta}(k)\Big(\sum_{i=1}^{N}C_{i\eta}^{T}R_i(j)D_{i\eta}^{T}\Big)\Big]\\
&\quad=\sum_{\eta=1}^{p} tr\Big[P_{\eta}^{T}(k)\Big(\sum_{i=1}^{N}A_{i\eta}^{T}R_i(j)B_{i\eta}^{T}\Big)+P_{\eta}^{T}(k)\Big(\sum_{i=1}^{N}D_{i\eta}R_i^{T}(j)C_{i\eta}\Big)\Big]\\
&\quad=\sum_{\eta=1}^{p} tr\Big[P_{\eta}^{T}(k)\sum_{i=1}^{N}\left(A_{i\eta}^{T}R_i(j)B_{i\eta}^{T}+D_{i\eta}R_i^{T}(j)C_{i\eta}\right)\Big]\\
&\quad=\sum_{\eta=1}^{p} tr\left[P_{\eta}^{T}(k)\left(P_{\eta}(j)-\frac{\sum_{\omega=1}^{N}\|R_{\omega}(j)\|^2}{\sum_{\omega=1}^{N}\|R_{\omega}(j-1)\|^2}P_{\eta}(j-1)\right)\right]\\
&\quad=\sum_{\eta=1}^{p} tr\Big[P_{\eta}^{T}(k)P_{\eta}(j)\Big]-\frac{\sum_{\omega=1}^{N}\|R_{\omega}(j)\|^2}{\sum_{\omega=1}^{N}\|R_{\omega}(j-1)\|^2}\\
&\qquad\times\sum_{\eta=1}^{p} tr[P_{\eta}^{T}(k)P_{\eta}(j-1))\Big].
\end{aligned}$$

Combining the preceding two relations gives the conclusion of this lemma. □

Lemma 2.2. Suppose that the sequences $\{P_{\eta}(k)\}, \eta\in I[1,p]$ and $\{R_i(k)\}, i\in I[1,N]$ are generated by algorithm 2.1. If there exists an integer number $\varphi\geq 1$, such that $\sum_{i=1}^{N}|R_i(k)||^2\neq 0$, for all $k=0,1,\ldots,\varphi$, then

$$\sum_{i=1}^{N} tr[R_i^{T}(k)R_i(j)]=0, \quad \sum_{\eta=1}^{p} tr[P_{\eta}^{T}(j)P_{\eta}(k)]=0. \tag{2.1.3}$$

for all $j,k=1,\ldots,\varphi, j\neq k$.

Proof. It is known that

$$tr[R_i^{T}(k)R_i(j)]=tr[R_i^{T}(j)R_i(k)], \quad tr[P_{\eta}^{T}(k)P_{\eta}(j)]=tr[P_{\eta}^{T}(j)P_{\eta}(k)].$$

We only need to indicate that

$$\sum_{i=1}^{N} tr[R_i^{T}(k)R_i(j)]=0, \quad \sum_{\eta=1}^{p} tr[P_{\eta}^{T}(k)P_{\eta}(j)]=0. \tag{2.1.4}$$

for $1\leq j<k\leq\varphi$.

We apply induction to prove conclusion (2.1.4), and also we do it in two steps. Step 1. For $j=1$, by using lemma 2.1 we can obtain

$$\begin{aligned}
&\sum_{i=1}^{N} tr[R_i^T(2)R_i(1)]\\
&\quad=\sum_{i=1}^{N} tr[R_i^T(1)R_i(1)]-\frac{\sum_{\omega=1}^{N}\|R_\omega(1)\|^2}{\sum_{\tau=1}^{p}\|P_\tau(1)\|^2}\sum_{\eta=1}^{p} tr[P_\eta^T(1)P_\eta(1)]\\
&\quad=\sum_{i=1}^{N}\|R_i(1)\|^2-\frac{\sum_{\omega=1}^{N}\|R_\omega(1)\|^2}{\sum_{\tau=1}^{p}\|P_\tau(1)\|^2}\sum_{\eta=1}^{p}\|P_\eta(1)\|^2=0.
\end{aligned}$$

And from algorithm 2.1 one also has

$$\begin{aligned}
&\sum_{\eta=1}^{p} tr\Big[P_\eta^T(2)P_\eta(1)\Big]\\
&\quad=\sum_{\eta=1}^{p} tr\Big[\Big(\sum_{i=1}^{N}(A_{i\eta}^T R_i(2)B_{i\eta}^T+D_{i\eta}R_i^T(2)C_{i\eta})\Big)^T P_\eta(1)\Big]\\
&\qquad+\frac{\sum_{\omega=1}^{N}\|R_\omega(2)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(1)\|^2}\sum_{\eta=1}^{p} tr[P_\eta^T(1)P_\eta(1)]\\
&\quad=\sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N}(B_{i\eta}R_i^T(2)A_{i\eta}+C_{i\eta}^T R_i(2)D_{i\eta}^T)P_\eta(1)\Big]\\
&\qquad+\frac{\sum_{\omega=1}^{N}\|R_\omega(2)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(1)\|^2}\sum_{\eta=1}^{p}\|P_\eta(1)\|^2\\
&\quad=\sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N}\Big(R_i^T(2)A_{i\eta}P_\eta(1)B_{i\eta}+R_i(2)D_{i\eta}^T P_\eta(1)C_{i\eta}^T\Big)\Big]\\
&\qquad+\frac{\sum_{\omega=1}^{N}\|R_\omega(2)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(1)\|^2}\sum_{\eta=1}^{p}\|P_\eta(1)\|^2\\
&\quad=\sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N}\Big(R_i^T(2)A_{i\eta}P_\eta(1)B_{i\eta}+R_i^T(2)C_{i\eta}P_\eta^T(1)D_{i\eta}\Big)\Big]\\
&\qquad+\frac{\sum_{\omega=1}^{N}\|R_\omega(2)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(1)\|^2}\sum_{\eta=1}^{p}\|P_\eta(1)\|^2\\
&\quad=\sum_{i=1}^{N} tr\Big[R_i^T(2)\sum_{\eta=1}^{p}\Big(A_{i\eta}P_\eta(1)B_{i\eta}+C_{i\eta}P_\eta^T(1)D_{i\eta}\Big)\Big]\\
&\qquad+\frac{\sum_{\omega=1}^{N}\|R_\omega(2)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(1)\|^2}\sum_{\eta=1}^{p}\|P_\eta(1)\|^2\\
&\quad=\sum_{i=1}^{N} tr\Big[R_i^T(2)(R_i(1)-R_i(2))\Big]\frac{\sum_{\tau=1}^{p}\|P_\tau(1)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(1)\|^2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sum_{\omega=1}^{N}\|R_\omega(2)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(1)\|^2}\sum_{\eta=1}^{p}\|P_\eta(1)\|^2\\
&=-\sum_{i=1}^{N}\|R_i(2)\|^2\frac{\sum_{\tau=1}^{p}\|P_\tau(1)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(1)\|^2}+\frac{\sum_{\omega=1}^{N}\|R_\omega(2)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(1)\|^2}\sum_{\eta=1}^{p}\|P_\eta(1)\|^2=0.
\end{aligned}
$$

Step 2. In this step, it is assumed that

$$
\sum_{i=1}^{N} tr[R_i^T(v)R_i(j)]=0,\quad \sum_{\eta=1}^{p} tr[P_\eta^T(v)P_\eta(j)]=0, \tag{2.1.5}
$$

Now we prove

$$
\sum_{i=1}^{N} tr[R_i^T(v+1)R_i(j)]=0,\quad \sum_{\eta=1}^{p} tr[P_\eta^T(v+1)P_\eta(j)]=0, \tag{2.1.6}
$$

for $j=1,2,\ldots,v$.

By applying lemma 2.1 and step 1, for $j=1,2,\ldots,v-1$, it follows from the induction assumption (2.1.5),

$$
\begin{aligned}
&\sum_{i=1}^{N} tr[R_i^T(v+1)R_i(j)]\\
&=\sum_{i=1}^{N} tr[R_i^T(v)R_i(j)]-\frac{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}{\sum_{\tau=1}^{p}\|P_\tau(v)\|^2}\sum_{\eta=1}^{p} tr[P_\eta^T(v)P_\eta(j)]\\
&\quad+\frac{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2\sum_{\omega=1}^{N}\|R_\omega(j)\|^2}{\sum_{\tau=1}^{p}\|P_\tau(v)\|^2\sum_{\omega=1}^{N}\|R_\omega(j-1)\|^2}\times\sum_{\eta=1}^{p} tr[P_\eta^T(v)P_\eta(j-1)]=0.
\end{aligned}
$$

In addition, it can be obtained from lemma 2.1 and induction assumption (2.1.5) that

$$
\begin{aligned}
&\sum_{i=1}^{N} tr[R_i^T(v+1)R_i(v)]\\
&=\sum_{i=1}^{N}\|R_i(v)\|^2-\frac{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}{\sum_{\tau=1}^{p}\|P_\tau(v)\|^2}\sum_{\eta=1}^{p}\|P_\eta(v)\|^2\\
&\quad+\frac{\left[\sum_{\omega=1}^{N}\|R_\omega(v)\|^2\right]^2}{\sum_{\tau=1}^{p}\|P_\tau(v)\|^2\sum_{\omega=1}^{N}\|R_\omega(v-1)\|^2}\sum_{\eta=1}^{p} tr[P_\eta^T(v)P_\eta(v-1)]\\
&=\frac{\left[\sum_{\omega=1}^{N}\|R_\omega(v)\|^2\right]^2}{\sum_{\tau=1}^{p}\|P_\tau(v)\|^2\sum\limits_{\omega=1}^{N}\|R_\omega(v-1)\|^2}\sum_{\eta=1}^{p} tr[P_\eta^T(v)P_\eta(v-1)]=0.
\end{aligned}
$$

It shows that the second relation in (2.1.6) holds for $j = 1, 2, \ldots, v$. From algorithm 2.1, it can also be obtained that

$$\begin{aligned}
&\sum_{\eta=1}^{p} tr\Big[P_\eta^T(v+1)P_\eta(j)\Big] \\
&\quad = \sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N}\Big(A_{i\eta}^T R_i(v+1)B_{i\eta}^T + D_{i\eta}R_i^T(v+1)C_{i\eta}\Big)^T P_\eta(j)\Big] \\
&\qquad + \frac{\sum_{\omega=1}^{N}\|R_\omega(v+1)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}\sum_{\eta=1}^{p} tr\Big[P_\eta^T(v)P_\eta(j)\Big] \\
&\quad = \sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N}\Big(B_{i\eta}R_i^T(v+1)A_{i\eta} + C_{i\eta}^T R_i(v+1)D_{i\eta}^T\Big)P_\eta(j)\Big] \\
&\qquad + \frac{\sum_{\omega=1}^{N}\|R_\omega(v+1)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}\sum_{\eta=1}^{p} tr\Big[P_\eta^T(v)P_\eta(j)\Big] \\
&\quad = \sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N}\Big(R_i^T(v+1)A_{i\eta}P_\eta(j)B_{i\eta} + R_i(v+1)D_{i\eta}^T P_\eta(j)C_{i\eta}^T\Big)\Big] \\
&\qquad + \frac{\sum_{\omega=1}^{N}\|R_\omega(v+1)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}\sum_{\eta=1}^{p} tr\Big[P_\eta^T(v)P_\eta(j)\Big] \\
&\quad = \sum_{\eta=1}^{p} tr\left[\sum_{i=1}^{N}\Big(R_i^T(v+1)A_{i\eta}P_\eta(j)B_{i\eta} + R_i^T(v+1)C_{i\eta}P_\eta^T(j)D_{i\eta}\Big)\right] \\
&\qquad + \frac{\sum_{\omega=1}^{N}\|R_\omega(v+1)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}\sum_{\eta=1}^{p} tr\Big[P_\eta^T(v)P_\eta(j)\Big] \\
&\quad = \sum_{\eta=1}^{p} tr\Big[R_i^T(v+1)\sum_{i=1}^{N}\Big(A_{i\eta}P_\eta(j)B_{i\eta} + C_{i\eta}P_\eta^T(j)D_{i\eta}\Big)\Big] \\
&\qquad + \frac{\sum_{\omega=1}^{N}\|R_\omega(v+1)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}\sum_{\eta=1}^{p} tr\Big[P_\eta^T(v)P_\eta(j)\Big] \\
&\quad = \frac{\sum_{\tau=1}^{p}\|P_\tau(j)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(j)\|^2}\sum_{i=1}^{N} tr\Big[R_i^T(v+1)\Big(R_i(j) - R_i(j+1)\Big)\Big] \\
&\qquad + \frac{\sum_{\omega=1}^{N}\|R_\omega(v+1)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}\sum_{i=1}^{N} tr\Big[P_\eta^T(v)P_\eta(j)\Big].
\end{aligned}$$

From this relation, it follows from the induction (2.1.5) that the first relation in (2.1.6), for $j = 1, 2, \ldots, v-1$, can be written as

$$\sum_{\eta=1}^{p} tr[P_\eta^T(v+1)P_\eta(j) = 0.$$

For $j = v$, from the above relation we have

$$\begin{aligned}
&\sum_{\eta=1}^{p} tr\Big[P_\eta^T(v+1)P_\eta(v)\Big]\\
&\quad= \frac{\sum_{\tau=1}^{p}\|P_\tau(v)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}\Big(\sum_{i=1}^{N} t\Big[R_i^T(v+1)R_i(v)\Big]\\
&\qquad-\sum_{i=1}^{N} tr\Big[R_i^T(v+1)R_i(v+1)\Big]\Big)\\
&\qquad+\frac{\sum_{\omega=1}^{N}\|R_\omega(v+1)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}\sum_{i=1}^{N} tr\Big[P_\eta^T(v)P_\eta(v)\Big]\\
&\quad= -\frac{\sum_{\tau=1}^{p}\|P_\tau(v)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}\sum_{i=1}^{N}\|R_i(v+1)\|^2\\
&\qquad+\frac{\sum_{\omega=1}^{N}\|R_\omega(v+1)\|^2}{\sum_{\omega=1}^{N}\|R_\omega(v)\|^2}\sum_{\eta=1}^{p}\|P_\eta(v)\|^2 = 0.
\end{aligned}$$

Thus the second relation in (2.1.6) holds for $j = 1, 2, \ldots, v$.

From steps 1 and 2, the conclusion is immediately obtained by the principle of induction. □

Lemma 2.3. Assume that the coupled Sylvester-transpose matrix equation (2.1.1) are consistent and let $[X_{1*}, X_{2*}, \ldots, X_{p*}]$ be their solutions. Then, for any initial matrices $X_\eta(1), \eta \in I[1,p]$, the sequences $\{X_\eta(k)\}, \{P_\eta(k)\}, \eta \in I[1,p]$ and $\{R_i(k)\}$, $i \in I[1,N]$, generated by algorithm 2.1 satisfy

$$\sum_{\eta=1}^{p} tr[P_\eta^T(k)(X_{\eta*} - X_\eta(k))] = \sum_{i=1}^{N}\|R_i(k)\|^2, \quad k = 1, 2, \ldots \tag{2.1.7}$$

Proof. We prove the conclusion by induction. For $i = 1$, we have

$$\begin{aligned}
&\sum_{\eta=1}^{p} tr\Big[P_\eta^T(1)\Big(X_{\eta*} - X_\eta(1)\Big)\Big]\\
&\quad= \sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N}(B_{i\eta}R_i^T(1)A_{i\eta} + C_{i\eta}^T R_i(1)D_{i\eta}^T)(X_{\eta*} - X_\eta(1))\Big]\\
&\quad= \sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N}\Big(R_i^T A_{i\eta}(X_{\eta*} - X_\eta(1))B_{i\eta}\\
&\qquad+ R_i(1)D_{i\eta}^T(X_{\eta*} - X_\eta(1))C_{i\eta}^T\Big)\Big]\\
&\quad= \sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N}\Big(R_i^T A_{i\eta}(X_{\eta*} - X_\eta(1))B_{i\eta}\\
&\qquad+ R_i^T(1)C_{i\eta}(X_{\eta*} - X_\eta(1))^T D_{i\eta}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} tr\Big[R_i^T \sum_{\eta=1}^{p} \Big(A_{i\eta}(X_{\eta *} - X_\eta(1))B_{i\eta} \\
&\quad + C_{i\eta}(X_{\eta *} - X_\eta(1))^T D_{i\eta}\Big)\Big] \\
&= \sum_{i=1}^{N} tr\Big[R_i^T \Big(F_i - \sum_{\eta=1}^{p}(A_{i\eta} X_\eta(1))B_{i\eta} + C_{i\eta} X_\eta^T(1) D_{i\eta})\Big)\Big] \\
&= \sum_{i=1}^{N} tr\Big[R_i^T R_i\Big] \\
&= \sum_{i=1}^{N} \|R_i\|^2.
\end{aligned}
$$

Now assume the conclusion (2.1.7) holds for $k = n$. Then we can get

$$
\sum_{\eta=1}^{p} tr[P_\eta^T(n)(X_{\eta *} - X_\eta(n))] = \sum_{i=1}^{N} \|R_i(n)\|^2.
$$

It follows from algorithm 2.1 that

$$
\begin{aligned}
&\sum_{\eta=1}^{p} tr\Big[P_\eta^T(n+1)\Big(X_{\eta *} - X_\eta(n+1)\Big)\Big] \\
&\quad = \sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N} \Big(B_{i\eta} R_i^T(n+1) A_{i\eta} + C_{i\eta}^T R_i(n+1) D_{i\eta}^T\Big) \\
&\qquad \times \Big(X_{\eta *} - X_\eta(n+1)\Big)\Big] \\
&\qquad + \frac{\sum_{\omega=1}^{N} \|R_\omega(n)\|^2}{\sum_{\omega=1}^{N} \|R_\omega(n)\|^2} tr\Big[\sum_{\eta=1}^{p} P_\eta^T(n)\Big(X_{\eta *} - X_\eta(n)\Big)\Big]
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
&\sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N} \Big(B_{i\eta} R_i^T(n+1) A_{i\eta} + C_{i\eta}^T R_i(n+1) D_{i\eta}^T\Big) \times \Big(X_{\eta *} - X_\eta(n+1)\Big)\Big] \\
&\quad = \sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N} \Big(R_i^T(n+1) A_{i\eta}(X_{\eta *} - X_\eta(n+1)\Big) B_{i\eta} \\
&\quad + R_i(n+1) D_{i\eta}^T\Big(X_{\eta *} - X_\eta(n+1)) C_{i\eta}^T\Big)\Big] \\
&\quad = \sum_{\eta=1}^{p} tr\Big[\sum_{i=1}^{N} \Big(R_i^T(n+1) A_{i\eta}(X_{\eta *} - X_\eta(n+1)\Big) B_{i\eta} \\
&\quad + R_i^T(n+1) C_{i\eta}\Big(X_{\eta *} - X_\eta(n+1))^T D_{i\eta}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} tr\Big[\sum_{\eta=1}^{p} R_i^T(n+1)\Big(A_{i\eta}\Big(X_{\eta*} - X_\eta(n+1)\Big)B_{i\eta} \\
&\quad + C_{i\eta}\Big(X_{\eta*} - X_\eta(n+1)\Big)^T D_{i\eta}\Big)\Big] \\
&= \sum_{i=1}^{N} tr\Big[R_i^T(n+1)\Big(F_i - \sum_{\eta=1}^{p}\Big(A_{i\eta}X_\eta(n+1)B_{i\eta} \\
&\quad + C_{i\eta}X_\eta^T(n+1)D_{i\eta}\Big)\Big)\Big] \\
&= \sum_{i=1}^{N} \|R_i(n+1)\|^2.
\end{aligned}
$$

By induction and algorithm 2.1 we can obtain

$$
\begin{aligned}
&\sum_{\eta=1}^{p} tr\Big[P_\eta^T(n)(X_{\eta*} - X_\eta(n+1))\Big] \\
&\quad = tr\Big[\sum_{\eta=1}^{p} P_\eta^T(n)\Big(X_{\eta*} - X_\eta(n) - \frac{\sum_{\omega=1}^{N}\|R_\omega(n)\|^2}{\sum_{\tau=1}^{p}\|P_\tau(n)\|^2}P_\eta(n)\Big)\Big] \\
&\quad = tr\Big[\sum_{\eta=1}^{p} P_\eta^T(n)(X_{\eta*} - X_\eta(n))\Big] \\
&\qquad - \frac{\sum_{\omega=1}^{N}\|R_\omega(n)\|^2}{\sum_{\tau=1}^{p}\|P_\tau(n)\|^2}\sum_{\eta=1}^{p} tr\Big[P_\eta^T(n)P_\eta(n)\Big] \\
&\quad = \sum_{i=1}^{N}\|R_i(n))\|^2 - \sum_{\omega=1}^{N}\|R_\omega(n)\|^2 = 0.
\end{aligned}
$$

Thus we have

$$\sum_{\eta=1}^{p} tr[P_\eta^T(n+1)(X_{\eta*} - X_\eta(n+1))] = \sum_{i=1}^{N}\|R_i(n+1)\|^2.$$

This implies that (2.1.7) holds for $k = n+1$. Thus, we complete the proof. □

Lemma 2.4. [23]Let $P(m,n) \in \Re^{mn\times mn}$ be a square $mn \times mn$ matrix partitioned into $m \times n$ sub-matrices such that jith position be 1 and zeros elsewhere, *i.e.*

$$P(m,n) = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij} \otimes E_{ij}^T, \tag{2.1.8}$$

where $E_{ij} = e_i e_j^T$ called an elementary matrix of order $m \times 1(n \times 1)$. Using this definition, we have

$$P(m,n)\mathrm{vec}(X^T)=\mathrm{vec}(X),\quad P(m,n)P(n,m)=I_{mn},$$
$$P^T(m,n)=P^{-1}(m,n)=P(n,m).$$

Lemma 2.5. [28]Suppose that the consistent system of linear equation $Ax=b$ has a solution $x^*\in R(A^H)$, then x^* is the unique least norm solution of the system of linear equation.

Theorem 2.1. Assume that the coupled Sylvester-transpose matrix equations (2.1.1) are consistent, then for any initial matrices $(X_1,X_2,\ldots,X_p)$, the sequences $X_i(k), i\in I[1,p]$ given by the algorithm 2.1, convergence to their solutions with finite iterative steps in the absence of roundoff errors. Furthermore, if we take the initial matrices

$$X_i(1)=\sum_{\eta=1}^{N}(A_{\eta i}^TE_\eta B_{\eta i}^T+D_{\eta i}E_\eta^TC_{\eta i}),\quad i=1,2,\ldots,p, \tag{2.1.9}$$

where $E_\eta, \eta=1,2,\ldots,N$, are arbitrary matrices, or especially $X_i(1)=0, i=1,2,\ldots,p$, then the solution group $(X_{1*},X_{2*},\ldots,X_{p*})$, generated by algorithm 2.1, is the least Frobinius norm solution group of the coupled matrix equations (2.1.1).

Proof. Firstly, in the space $\mathfrak{R}^{m_1\times p_1}\times\mathfrak{R}^{m_2\times p_2}\times\cdots\times\mathfrak{R}^{m_N\times p_N}$, we define an inner product as

$$\langle(A_1,A_2,\ldots,A_N),(B_1,B_2,\cdots,B_N)\rangle=\sum_{i=1}^{N}tr(B_i^TA_i),$$

for $(A_1,A_2,\ldots,A_N),(B_1,B_2,\ldots,B_N)\in\mathfrak{R}^{m_1\times p_1}\times\mathfrak{R}^{m_2\times p_2}\times\cdots\times\mathfrak{R}^{m_N\times p_N}$.

If $\sum_{i=1}^{N}\|R_i(k)\|^2\neq0, k=1,2,\ldots,q=2\sum_{i=1}^{N}m_ip_i$, then from previous lemmas we get $\sum_{j=1}^{p}\|P_j(k)\|^2\neq0$. Now we can compute $R_i(q+1)$ and $[X_1(q+1), X_2(q+1),\ldots,X_p(q+1)]$.

From lemma 2.2, it is not difficult to get

$$\sum_{i=1}^{N}(R_i^T(q+1)R_i(j))=0, j=1,2,\ldots,q,$$

and

$$\sum_{i=1}^{N}(R_i^T(m)R_i(j))=0, m,j=1,2,\ldots,q, m\neq j.$$

Then $[R_1(k),R_2(k),\ldots,R_N(k)], k=1,2,\ldots,2q$, is a group of orthogonal basis of the previously defined inner product space. It follows from $[R_1(q+1), R_2(q+1),\ldots,R_N(q+1)]=0$ that $[X_1(q+1),X_2(q+1),\ldots,X_p(q+1)]=0$ is a solution group of problem 2.1. Therefore when problem 2.1 is consistent, we can verify that the solution group of problem 2.1 can be obtained within finite iterative steps. □

Now let E_η be arbitrary matrices for $\eta = 1, 2, \ldots, N$. We can get

$$\begin{pmatrix} \operatorname{vec}\Big(\sum_{\eta=1}^{N}(A_{\eta1}^T E_\eta B_{\eta1}^T + D_{\eta1} E_\eta^T C_{\eta1})\Big) \\ \operatorname{vec}\Big(\sum_{\eta=1}^{N}(A_{\eta2}^T E_\eta B_{\eta2}^T + D_{\eta2} E_\eta^T C_{\eta2})\Big) \\ \vdots \\ \operatorname{vec}\Big(\sum_{\eta=1}^{N}(A_{\eta p}^T E_\eta B_{\eta p}^T + D_{\eta p} E_\eta^T C_{\eta p})\Big) \end{pmatrix}$$

$$= \begin{pmatrix} B_{11}\otimes A_{11}^T + (C_{11}^T \otimes D_{11})P(m_1, p_1) & \ldots & B_{N1}\otimes A_{N1}^T + (C_{N1}^T \otimes D_{N1})P(m_N, p_N) \\ B_{12}\otimes A_{12}^T + (C_{12}^T \otimes D_{12})P(m_1, p_1) & \ldots & B_{N2}\otimes A_{N2}^T + (C_{N2}^T \otimes D_{N2})P(m_N, p_N) \\ \vdots & \vdots & \vdots \\ B_{1p}\otimes A_{1p}^T + (C_{1p}^T \otimes D_{1p})P(m_1, p_1) & \ldots & B_{Np}\otimes A_{Np}^T + (C_{Np}^T \otimes D_{Np})P(m_N, p_N) \end{pmatrix} \times \begin{pmatrix} \operatorname{vec}(E_1) \\ \operatorname{vec}(E_2) \\ \vdots \\ \operatorname{vec}(E_N) \end{pmatrix}$$

$$= \begin{pmatrix} B_{11}^T\otimes A_{11} + P^T(m_1, p_1)(C_{11} \otimes D_{11}^T) & \ldots & B_{1p}^T\otimes A_{1p} + P^T(m_1, p_1)(C_{1p} \otimes D_{1p}^T) \\ B_{21}^T\otimes A_{21} + P^T(m_2, p_2)(C_{21} \otimes D_{21}^T) & \ldots & B_{2p}^T\otimes A_{2p} + P^T(m_2, p_2)(C_{2p} \otimes D_{2p}^T) \\ \vdots & \vdots & \vdots \\ B_{N1}^T\otimes A_{N1} + P^T(m_N, p_N)(C_{N1} \otimes D_{N1}^T) & \ldots & B_{Np}^T\otimes A_{Np} + P^T(m_N, p_N)(C_{Np} \otimes D_{Np}^T) \end{pmatrix} \times \begin{pmatrix} \operatorname{vec}(E_1) \\ \operatorname{vec}(E_2) \\ \vdots \\ \operatorname{vec}(E_N) \end{pmatrix}$$

$$\in R\left(\begin{pmatrix} B_{11}^T\otimes A_{11} + P^T(m_1, p_1)(C_{11} \otimes D_{11}^T) & \ldots & B_{1p}^T\otimes A_{1p} + P^T(m_1, p_1)(C_{1p} \otimes D_{1p}^T) \\ B_{21}^T\otimes A_{21} + P^T(m_2, p_2)(C_{21} \otimes D_{21}^T) & \ldots & B_{2p}^T\otimes A_{2p} + P^T(m_2, p_2)(C_{2p} \otimes D_{2p}^T) \\ \vdots & \vdots & \vdots \\ B_{N1}^T\otimes A_{N1} + P^T(m_N, p_N)(C_{N1} \otimes D_{N1}^T) & \ldots & B_{Np}^T\otimes A_{Np} + P^T(m_N, p_N)(C_{Np} \otimes D_{Np}^T) \end{pmatrix}^T\right).$$

If we consider the initial matrices

$$X_i(1)=\sum_{\eta=1}^{N}(A_{\eta i}^TE_\eta B_{\eta i}^T+D_{\eta i}E_\eta^TC_{\eta i}),\, i\in[1,p],$$

where $E_\eta, \eta=1,2,\ldots,N$, are the arbitrary matrices, then all $X_i(k)$ for $i=1,2,\ldots,p$, generated by algorithm 2.1 satisfy

$$\begin{pmatrix}\mathrm{vec}(X_1(k))\\ \mathrm{vec}(X_2(k))\\ \vdots\\ \mathrm{vec}(X_p(k))\end{pmatrix}\in R\left(\begin{pmatrix} B_{11}^T\otimes A_{11}+P^T(m_1,p_1)(C_{11}\otimes D_{11}^T) & \ldots & B_{1p}^T\otimes A_{1p}+P^T(m_1,p_1)(C_{1p}\otimes D_{1p}^T)\\ B_{21}^T\otimes A_{21}+P^T(m_2,p_2)(C_{21}\otimes D_{21}^T) & \ldots & B_{2p}^T\otimes A_{2p}+P^T(m_2,p_2)(C_{2p}\otimes D_{2p}^T)\\ \vdots & \vdots & \vdots\\ B_{N1}^T\otimes A_{N1}+P^T(m_N,p_N)(C_{N1}\otimes D_{N1}^T) & \ldots & B_{Np}^T\otimes A_{Np}+P^T(m_N,p_N)(C_{Np}\otimes D_{Np}^T)\end{pmatrix}^T\right).$$

Hence, with the initial matrix group $(X_1(1),X_2(1),\ldots,X_p(1))$ where

$$X_i(1)=\sum_{\eta=1}^{N}(A_{\eta i}^TE_\eta B_{\eta i}^T+D_{\eta i}E_\eta^TC_{\eta i}),\, i\in[1,p],$$

according to lemma 2.5, the solution group $(X_{1*}(1),X_{2*}(1),\ldots,X_{p*}(1))$ obtained by algorithm 2.1 is the least Frobenius norm solution group.

The proof of problem 2.2 will be given as follows.

When the matrix equations (2.1.1) are consistent, the solution set S_E of the coupled matrix equations (2.1.1) is non-empty. For a given matrix group $(\widehat{X}_1,\widehat{X}_2,\ldots,\widehat{X}_p)$ with $\widehat{X}_j\in R^{l_j\times n_j}, j=1,2,\ldots,p$, it is not difficult to get

$$\begin{aligned}&\sum_{\eta=1}^{N}(A_{i\eta}X_\eta B_{i\eta}+C_{i\eta}X_\eta^TD_{i\eta})=F_i,\quad i=1,2,\ldots,N,\\ &\iff\sum_{\eta=1}^{N}(A_{i\eta}(X_\eta-\widehat{X}_\eta)B_{i\eta}+C_{i\eta}(X_\eta-\widehat{X}_\eta)^TD_{i\eta})\\ &=F_i-\sum_{\eta=1}^{N}(A_{i\eta}\widehat{X}_\eta B_{i\eta}+C_{i\eta}\widehat{X}_\eta^TD_{i\eta}),\, i=1,2,\ldots,N.\end{aligned}$$

Define $\overline{X}_\eta=X_\eta-\widehat{X}_\eta$ and $\overline{F}_i=F_i-\sum_{\eta=1}^{N}(A_{i\eta}\widehat{X}_\eta B_{i\eta}+C_{i\eta}\widehat{X}_\eta^TD_{i\eta})$ for $i=1,2,\ldots,N$. Then the matrix nearness problem 2.1 is equivalent to first finding the least Frobenius norm solution group $(\overline{X}_{1*},\overline{X}_{2*},\ldots,\overline{X}_{p*})$ of new coupled Sylvester-transpose matrix equations

$$\sum_{\eta=1}^{N}(A_{i\eta}\overline{X}_{\eta}B_{i\eta}+C_{i\eta}\overline{X}_{\eta}^{T}D_{i\eta})=\overline{F}_{i},\quad i=1,2,\ldots,N,$$

which can be obtained by using algorithm 2.1 with the initial matrices

$$\overline{X}_{i}(1)=\sum_{\eta=1}^{N}(A_{\eta i}^{T}E_{\eta}B_{\eta i}^{T}+D_{\eta i}E_{\eta}^{T}C_{\eta i}),\, i\in[1,p],$$

where E_η are arbitrary matrices, or especially, $\overline{X}_i(1)=0$ for $i=1,2,\ldots,p$. Here the matrix solution group of the matrix nearness problem 2.2 can be represented as $(\widetilde{X}_1,\widetilde{X}_2,\ldots,\widetilde{X}_p)$ with

$$\widetilde{X}_{\eta}=\overline{X}_{\eta *}+\widehat{X}_{\eta},\quad \eta=1,2,\ldots,p.$$

Thus, the proof of problem 2.2 can be finished. □

2.2 Numerical Example

In this section, we report a numerical example to test the proposed iterative method.

Example 2.1. Consider the following coupled Sylvester-transpose matrix equations

$$\begin{cases} A_{11}XB_{11}+C_{11}X^{T}D_{11}+A_{12}YB_{12}+C_{12}Y^{T}D_{12}=E_{1}, \\ A_{21}XB_{21}+C_{21}X^{T}D_{21}+C_{22}Y^{T}D_{22}=E_{2}, \end{cases} \tag{2.2.1}$$

with the following parametric matrices

$$A_{11}=\begin{pmatrix}-1&2&1&1\\2&2&3&1\\2&1&2&2\\1&2&5&1\end{pmatrix},B_{11}=\begin{pmatrix}1&2&1&3\\5&2&4&1\\1&1&1&4\\2&6&2&1\end{pmatrix},C_{21}=\begin{pmatrix}21&-2&11&1\\2&-2&-3&1\\2&13&2&2\\11&32&5&1\end{pmatrix},$$

$$A_{12}=\begin{pmatrix}1&4&2&6\\2&2&6&1\\3&2&1&6\\2&3&5&2\end{pmatrix},A_{21}=\begin{pmatrix}1&1&1&1\\2&2&2&4\\1&5&3&1\\1&1&1&0\end{pmatrix},D_{21}=\begin{pmatrix}1&22&12&53\\5&2&-4&1\\-1&1&-1&4\\-2&6&2&1\end{pmatrix},$$

$$B_{21}=\begin{pmatrix}4&3&2&2\\4&1&2&1\\2&3&3&1\\4&2&2&0\end{pmatrix},C_{11}=\begin{pmatrix}0&2&1&1\\0&-5&3&1\\6&1&7&2\\2&2&4&1\end{pmatrix},D_{11}=\begin{pmatrix}-2&2&11&3\\-2&2&4&1\\1&1&0&0\\2&6&2&-8\end{pmatrix},$$

$$B_{12}=\begin{pmatrix}2&1&3&1\\4&3&2&6\\4&2&3&1\\1&2&3&0\end{pmatrix}, C_{22}=\begin{pmatrix}0&2&0&1\\0&11&21&1\\4&0&2&1\\0&-1&-1&1\end{pmatrix}, D_{22}=\begin{pmatrix}-3&12&11&2\\21&23&1&2\\1&2&4&1\\2&11&3&2\end{pmatrix},$$

$$E_1=\begin{pmatrix}1889&359&1384&1269\\543&396&513&507\\344&600&671&143\\387&535&698&354\end{pmatrix}, E_2=\begin{pmatrix}360&656&-138&340\\1201&1903&671&-284\\527&1539&555&1953\\304&2197&474&3999\end{pmatrix},$$

$$C_{12}=\begin{pmatrix}13&2&14&61\\2&12&0&1\\2&1&2&0\\1&2&6&1\end{pmatrix}, D_{12}=\begin{pmatrix}1&12&11&13\\5&2&4&1\\10&1&1&4\\-2&-6&2&1\end{pmatrix}.$$

It can be verified that the generalized coupled matrix equations (2.2.1) are consistent and have a unique solution pair (X^*, Y^*) as follows:

$$X^*=\begin{pmatrix}-1&2&2&1\\2&2&2&2\\2&2&3&2\\1&2&2&4\end{pmatrix}, Y^*=\begin{pmatrix}1&1&1&0\\1&2&1&1\\1&1&0&2\\1&1&2&1\end{pmatrix}.$$

If we let the initial matrix $(X(1), Y(1)) = 0$, applying algorithm 2.1, we obtain the solutions, which are

$$X(109)=\begin{pmatrix}-1.0000&2.0000&2.0000&1.0000\\1.9999&2.0000&1.9999&1.9999\\2.0000&1.9999&3.0000&2.0000\\0.9999&1.9999&2.0000&4.0000\end{pmatrix},$$

$$Y(109)=\begin{pmatrix}1.0000&0.9999&0.9999&0.0000\\1.0000&2.0000&0.9999&0.9999\\0.9999&1.0000&0.0000&2.0000\\0.9999&1.0000&2.0000&1.0000\end{pmatrix},$$

with corresponding residual

$$\|R(109)\| = 4.2982\times 10^{-11}.$$

The obtained results are presented in figure 2.1.1, where

$$e_k=\frac{\|(X(k), Y(k)-(X^*, Y^*)\|}{\|(X^*, X^*)\|}, \quad r_k=\|R(k)\|.$$

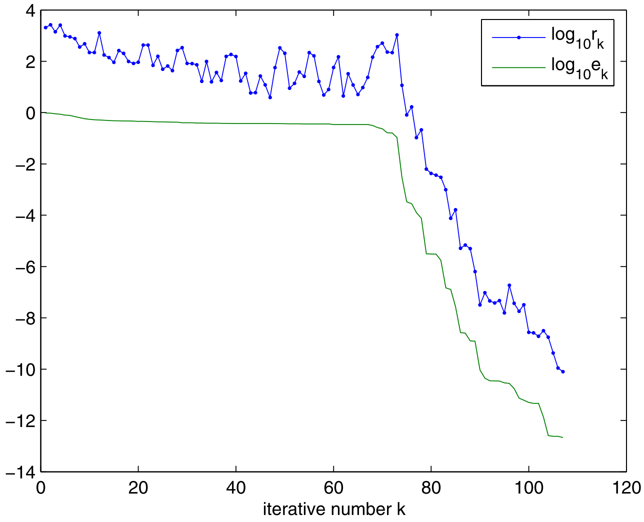

FIG. 2.2.1 – The relative error of solution and the residual for example 2.1.

Now let

$$\widehat{X}=\begin{pmatrix}1&1&1&0\\0&0&0&1\\1&7&1&1\\1&1&2&8\end{pmatrix},\ \widehat{Y}=\begin{pmatrix}1&0&2&0\\1&1&2&0\\1&0&1&0\\1&0&0&2\end{pmatrix},$$

By algorithm 2.1 for the generalized coupled Sylvester matrix equation (2.2.1), with the initial matrix pair $(\overline{X}(1),\overline{Y}(1))=0$, we have the least Frobenius norm solution of the generalized coupled Sylvester matrix equation (2.2.1) by the following form:

$$\overline{X}^*=\overline{X}(107)=\begin{pmatrix}-2.0000&0.9999&1.0000&1.0000\\2.0000&1.9999&1.9999&1.0000\\0.9999&-4.9999&1.9999&1.0000\\0.0000&1.0000&0.0000&-4.0000\end{pmatrix},$$

$$\overline{Y}^*=\overline{Y}(107)\begin{pmatrix}0.0000&1.0000&-1.0000&0.0000\\0.0000&0.9999&-0.9999&0.9999\\0.0000&0.9999&-0.9999&1.9999\\0.0000&0.9999&2.0000&-1.0000\end{pmatrix},$$

with corresponding residual

$$\|R(107)\|=7.8716\times10^{-11}.$$

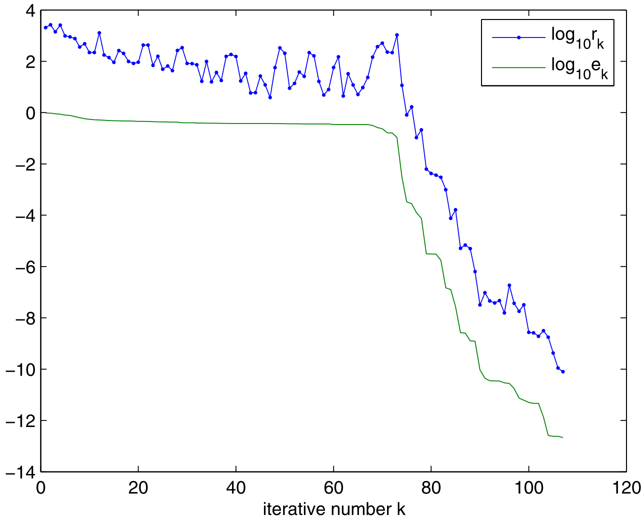

FIG. 2.2.2 – The relative error of least Frobenius norm solution and the residual for example 2.1.

The obtained results are presented in figure 2.2.2. Therefore, the solutions of the matrix equations nearness problem are

$$\widetilde{X} = \overline{X}^* + \widehat{X} = \begin{pmatrix} -1.0000 & 1.9999 & 2.0000 & 1.0000 \\ 2.0000 & 1.9999 & 1.9999 & 2.0000 \\ 1.9999 & 2.0001 & 2.9999 & 2.0000 \\ 1.0000 & 2.0000 & 2.0000 & 4.0000 \end{pmatrix},$$

$$\widetilde{Y} = \overline{Y}^* + \widehat{Y} = \begin{pmatrix} 1.0000 & 1.0000 & 1.0000 & 0.0000 \\ 1.0000 & 1.9999 & 1.0001 & 0.9999 \\ 1.0000 & 0.9999 & 0.0001 & 1.9999 \\ 1.0000 & 0.9999 & 2.0000 & 1.0000 \end{pmatrix}.$$

2.3 Control Application

A number of control problems are related to the SCA (state covariance assignment) problem[25], for example, controllability/obserbability Gramian assignment[26], a certain class of H_∞ control problems[27]. From a mathematical viewpoint, we point out that both the continuous-time and discrete-time SCA problems can be reduced to solving a symmetric matrix equation. Let us expand the continuous-time Lyapunov equation

$$A_K X + X A_K^T + B_K B_K^T = 0,$$

as

$$\begin{aligned}&(AX + XA^T + B_1B_1^T) + B_2K(C_2X + D_{21}B_1^T)\\&\quad + (C_2X + D_{21}B_1^T)^T K^T B_2^T + B_2K(D_{21}D_{21}^T)K^T B_2^T = 0.\end{aligned}$$

We also expand the discrete-time Lyapunov equation

$$A_K X A_K^T - X + B_K B_K^T = 0,$$

as

$$\begin{aligned}&(AXA^T - X + B_1B_1^T) + B_2K(C_2XA^T + D_{21}B_1^T)\\&\quad + (C_2XA^T + D_{21}B_1^T)^T K^T B_2^T + B_2K(C_2XC_2^T + D_{21}D_{21}^T)K^T B_2^T = 0.\end{aligned}$$

It is easy to see that both the continuous-time and discrete-time SCA problems are essentially the same from the viewpoint of mathematics.

We also consider the relationship between the matrix equation and some important special cases as a control problem. For example, let matrices $B \in \mathfrak{R}^{n\times m}$, $Q \in \mathfrak{R}^{n\times n}$, $R \in \mathfrak{R}^{p\times p}$ and $S \in \mathfrak{R}^{p\times n}$ be given where $Q = Q^T$, $R = R^T \geq 0$ and rank $R = r$. We focus on a symmetric matrix equation of quadratic type[26]:

$$Q + BKS + (BKS)^T + BKRK^T B^T = 0. \tag{2.3.1}$$

Thus, the solution of the symmetric matrix equation above is equivalent to the SCA problem with proper definitions of B, Q, R, and S. When $R = 0$, the special case of matrix equation above is reduced to the Karm-Yakubovich-transpose matrix equation $X - AX^T B = C$.

2.4 Conclusions

In this chapter, we have constructed an iterative algorithm for solving the generalized coupled Sylvester-transpose matrix equations. By this algorithm, the solvability of the generalized coupled Sylvester-transpose matrix equations can be derived automatically. Also when the considered coupled matrix equations are consistent, for any initial matrix group, a generalized solution group can be obtained in finite iterative steps in the absence of roundoff errors. Furthermore, we have solved problem 2.2 using the proposed algorithm 2.1. Finally, a numerical example is presented to support the theoretical results of this chapter.

Chapter 3

Finite Iterative Algorithm to Coupled Operator Matrix Equations with Sub-Matrix Constrained

In this chapter, we consider the following linear matrix equations

$$\Big[\sum_{i=1}^{p}\mathcal{A}_{1i}(X_i),\ \sum_{i=1}^{p}\mathcal{A}_{2i}(X_i),\ldots,\ \sum_{i=1}^{p}\mathcal{A}_{pi}(X_i)\Big] = \Big[M_1, M_2, \ldots, M_p\Big], \tag{3.0.1}$$

where $\mathcal{A}_{si} \in \mathfrak{LC}^{m_i\times n_i, p_s\times q_s}$ and $M_s \in \mathfrak{C}^{p_s\times q_s}$, $i, s = 1, 2, \ldots, p$. This coupled operator matrix equations (3.0.1) include the generalized coupled Sylvester matrix equations and generalized coupled Lyapunov matrix equations as special cases. In addition, all the constraint solutions in[17,19,20,23,34–46] can be summarized as the following cases.

(1) Symmetric (Skew-symmetric) constraint: $X = X^T(X = -X^T)$;
(2) Centrosymmetric (Centroskew symmetric) constraint: $X = S_nXS_n(-S_nXS_n)$;
(3) Reflexive (Anti-reflexive) constraint: $X = PXP(-PXP)$;
(4) Generalized reflexive (anti-reflexive) constraint: $X = PXQ(-PXQ)$;
(5) P orthogonal-symmetric (orthogonal-anti-symmetric) constraint: $(PX)^T = PX(-PX)$, where $P^T = P = P^{-1} \neq I_n$ and $Q^T = Q = Q^{-1} \neq I_n$.

It should be remarked that all the above common constraint solutions of matrix equations are all special cases of the following constraint

$$X = \mathcal{U}(X), \tag{3.0.2}$$

where $\mathcal{U} \in LC^{m\times n}$ is a self-conjugate involution operator, *i.e.*, $\mathcal{U}^* = \mathcal{U}, \mathcal{U}^2 = \mathcal{I}$.

From an operator point of view, in this chapter we construct a finite iterative algorithm to solve the coupled operator matrix equations (3.0.1) by extending the idea of conjugate gradient method. On the whole, we mainly consider the following two problems.

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Problem 3.1. Let $\mathcal{S}$ denote the set of constrained matrices defined by (3.0.2). For given $\mathcal{A}_{ij} \in LC^{m_j\times n_j, p_i\times q_i}$, $M_i \in C^{p_i\times q_i}$, $i, j = 1, 2, \ldots, p$, find $X_j \in \mathcal{S}, j = 1, 2, \ldots, p$, such that

$$\Big[\sum_{i=1}^{p}\mathcal{A}_{1i}(X_i), \sum_{i=1}^{p}\mathcal{A}_{2i}(X_i), \ldots, \sum_{i=1}^{p}\mathcal{A}_{pi}(X_i)] = [M_1, M_2, \ldots, M_p\Big]. \tag{3.0.3}$$

Problem 3.2. Let S denote the solution set of problem 3.1, for given $\widehat{X}_j \in \mathfrak{C}^{m_j\times n_j}, j = 1, 2, \ldots, p$, find $\overline{X}_j \in S, j = 1, 2, \ldots, p$, such that

$$\sum_{j=1}^{p}\|\overline{X_j} - \widehat{X}_j\|^2 = \min_{(X_1,\ldots,X_p)\in S}\left\{\sum_{j=1}^{p}\|X_j - \widehat{X}_j\|^2\right\}. \tag{3.0.4}$$

In the following, we define inner product as $\langle X, Y\rangle = tr(Y^H X)$ for all $X, Y \in \mathfrak{C}^{m\times n}$. If $\langle X, Y\rangle = 0$, two matrices X and Y are said to be orthogonal. In addition, let $LC^{m\times n, p\times q}$ denote the set of linear operators from $\mathfrak{C}^{m\times n}$ onto $\mathfrak{C}^{p\times q}$. Specially, when $p = m$ and $q = n$, $\mathfrak{LC}^{m\times n, p\times q}$ is denoted by $\mathfrak{LC}^{m\times n}$. For $\mathcal{A} \in \mathfrak{LC}^{m\times n, p\times q}$, its conjugate operator $\mathcal{A}^*$ is a linear operator satisfying the following equality

$$\langle \mathcal{A}(X), Y\rangle = \langle X, \mathcal{A}^*(Y)\rangle, \quad \text{for all } X \in \mathfrak{C}^{m\times n}, Y \in \mathfrak{C}^{p\times q}.$$

For example, if $\mathcal{A}$ is defined as $X \to AXB$, then we have $\mathcal{A}^* : Y \to A^H Y B^H$.

3.1 Iterative Method for Solving Problem 3.1

In this section, first of all, in order to solve problem 3.1, we propose an iterative algorithm. Next, some basic properties of the algorithm are presented. Finally, finding the least Frobenius norm solution to problem 3.1 is considered. In the sequel, the least norm solution always means the least Frobenius norm solution.

Algorithm 3.1

Step 1. Input $\mathcal{A}_{ij} \in \mathfrak{LC}^{m_j\times n_j, p_i\times q_i}$, $M_i \in \mathfrak{C}^{p_i\times q_i}$, and an arbitrary $X_j(1) \in \mathcal{S}$, $i, j = 1, 2, \ldots, p$;

Step 2. Compute

$$\begin{aligned}R(1) &= \operatorname{diag}\Big(M_1 - \sum_{i=1}^{p}\mathcal{A}_{1i}(X_i(1)), M_2 - \sum_{i=1}^{p}\mathcal{A}_{2i}(X_i(1)), \ldots, M_p - \sum_{i=1}^{p}\mathcal{A}_{pi}(X_i(1))\Big)\\ &= \operatorname{diag}(R_1(1), R_2(1), \ldots, R_p(1));\\ P_i(1) &= \frac{1}{2}\sum_{s=1}^{p}[\mathcal{A}_{si}^*(R_s(1)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(1))], i = 1, 2, \ldots, p; k := 0;\end{aligned}$$

Step 3. If $R(k)=0$ then stop and $(X_1(k), X_2(k), \ldots, X_p(k))$ is the constraint solution group;

else if $R(k) \neq 0$ but $P_m(k)=0$ for all $m=1,2,\ldots,p$, then stop and the general coupled matrix equations (3.0.1) are not consistent over constraint solution group;

else $k := k+1$;

Step 4. Compute

$$
\begin{aligned}
X_i(k) &= X_i(k-1) + \frac{\|R(k-1)\|_F{}^2}{\sum_{t=1}^{p} \|P_t(k-1)\|_F^2} P_i(k-1), \quad i=1,2,\ldots,p;\\
R(k) &= \operatorname{diag}\Big(M_1 - \sum_{i=1}^{p} \mathcal{A}_{1i}(X_i(k)), M_2 - \sum_{i=1}^{p} \mathcal{A}_{2i}(X_i(k)), \ldots, M_p - \sum_{i=1}^{p} \mathcal{A}_{\mathrm{pi}}(X_i(k))\Big)\\
&= R(k-1) - \frac{\|R(k-1)\|_F{}^2}{\sum_{t=1}^{p} \|P_i(k-1)\|_F^2} \times \operatorname{diag}\Big(\sum_{i=1}^{p} \mathcal{A}_{1i}(P_i(k-1)\Big),\\
&\quad \sum_{i=1}^{p} \mathcal{A}_{2i}(P_i(k-1)), \ldots, \sum_{i=1}^{p} \mathcal{A}_{pi}(P_i(k-1)))\\
&= \operatorname{diag}(R_1(k), R_2(k), \ldots, R_p(k));\\
Z_i(k) &= \frac{1}{2}\sum_{s=1}^{p} [\mathcal{A}_{si}^*(R_s(k)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(k))];\\
P_i(k) &= Z_i(k) + \frac{\|R(k)\|_F^2}{\|R(k-1)\|_F^2} P_i(k-1)\\
&= \frac{1}{2}\sum_{s=1}^{p} [\mathcal{A}_{si}^*(R_s(k)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(k))]\\
&\quad + \frac{\|R(k)\|_F^2}{\|R(k-1)\|_F^2} P_i(k-1), i=1,2,\ldots,p;
\end{aligned}
$$

Step 5. Go to step 3.

Some basic properties of algorithm 3.1 are listed in the following lemmas.

Lemma 3.1. The sequences $\{X_i(k)\}$ and $\{P_i(k)\}, i=1,2,\ldots,p$, generated by algorithm 3.1 are contained in the constraint set $\mathcal{S}$.

Proof. We prove the conclusion by induction. For $k=1, i=1,2,\ldots,p$, by $\mathcal{U}^2=\mathcal{I}$ and algorithm 3.1, we have

$$
\begin{aligned}
\mathcal{U}(P_i(1)) &= \mathcal{U}\Big(\frac{1}{2}\sum_{s=1}^{p} [\mathcal{A}_{si}^*(R_s(1)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(1))]\Big)\\
&= \frac{1}{2}\sum_{s=1}^{p} [\mathcal{U}\mathcal{A}_{si}^*(R_s(1)) + \mathcal{A}_{si}^*(R_s(1))] = P_i(1),
\end{aligned}
$$

i.e., $P_i(1) \in \mathcal{S}$.

For $k = 2$, since $X_i(1) \in \mathcal{S}, i = 1, 2, \ldots, p$, by algorithm 3.1, we have

$$\begin{aligned}\mathcal{U}(X_i(2)) &= \mathcal{U}(X_i(1)) + \frac{\|R(1)\|_F{}^2}{\sum_{t=1}^{p} \|P_i(1)\|_F^2}\mathcal{U}(P_i(1)) \\ &= X_i(1) + \frac{\|R(1)\|_F{}^2}{\sum_{t=1}^{p} \|P_i(1)\|_F^2} P_i(1) = X_i(2),\end{aligned}$$

i.e., $X_i(2) \in \mathcal{S}, i = 1, 2, \ldots, p$.

Moreover, we have

$$\begin{aligned}\mathcal{U}(P_i(2)) &= \mathcal{U}\Big(\frac{1}{2}\sum_{s=1}^{p}[\mathcal{A}_{si}^*(R_s(2)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(2))] + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2} P_i(1)\Big) \\ &= \mathcal{U}\Big(\frac{1}{2}\sum_{s=1}^{p}[\mathcal{A}_{si}^*(R_s(2)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(2))] + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2}\mathcal{U}(P_i(1))\Big) \\ &= \frac{1}{2}\sum_{s=1}^{p}[\mathcal{U}\mathcal{A}_{si}^*(R_s(2)) + \mathcal{A}_{si}^*(R_s(2))] + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2} P_i(1) \\ &= P_i(2),\end{aligned}$$

i.e., $P_i(2) \in \mathcal{S}, i = 1, 2, \ldots, p$.

Now if we assume that the conclusion holds for $k = m(m > 2)$, *i.e.* $X_i(m), P_i(m) \in \mathcal{S}, i = 1, 2, \ldots, p$, then we can obtain

$$\begin{aligned}&\mathcal{U}(X_i(m+1)) \\ &\quad = \mathcal{U}(X_i(m)) + \frac{\|R(m)\|_F{}^2}{\sum_{t=1}^{p} \|P_i(m)\|_F^2}\mathcal{U}(P_i(m)) \\ &\quad = X_i(m) + \frac{\|R(m)\|_F{}^2}{\sum_{t=1}^{p} \|P_i(m)\|_F^2} P_i(m) = X_i(m+1),\end{aligned}$$

and

$$\begin{aligned}&\mathcal{U}(P_i(m+1)) \\ &\quad = \mathcal{U}\left[\frac{1}{2}\sum_{s=1}^{p}(\mathcal{A}_{si}^*(R_s(m+1)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(m+1)))\right] + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2}\mathcal{U}(P_i(m)) \\ &\quad = \frac{1}{2}\sum_{s=1}^{p}[\mathcal{U}\mathcal{A}_{si}^*(R_s(m+1)) + \mathcal{A}_{si}^*(R_s^{(m+1)})] + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} P_i(m) \\ &\quad = P_i(m+1).\end{aligned}$$

Therefore, we can get

$$X_i(m+1), P_i(m+1) \in \mathcal{S}, i = 1, 2, \ldots, p.$$

Thus, by the principle of induction we draw the conclusion. □

Lemma 3.2. Supposed that $X^* = [X_1^*, X_2^*, \ldots, X_p^*]$ is a solution of problem 3.1. Let $X(k) = [X_1(k), X_2(k), \ldots, X_p(k)]$, then the sequences $\{X_i(k)\}, \{R(k)\}$ and $\{P(k)\}$ generated by algorithm 3.1 satisfy the following equality

$$\sum_{i=1}^{p} \langle X_i^* - X_i(k), P_i(k) \rangle = \|R(k)\|^2, \quad i = 1, 2, \ldots, p. \quad k = 1, 2, \ldots. \tag{3.1.1}$$

Proof. We prove the conclusion by induction. For $k = 1$, it follows from $X_i^*, X_i(1) \in \mathcal{S}$ that $X_i^* - X_i(1) \in \mathcal{S}$. Then by algorithm 3.1, we have

$$\begin{aligned}
&\sum_{i=1}^{p} \langle X_i^* - X_i(1), P_i(1) \rangle \\
&\quad = \sum_{i=1}^{p} \left\langle X_i^* - X_i(1), \frac{1}{2} \sum_{s=1}^{p} [\mathcal{A}_{si}^*(R_s(1)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(1))] \right\rangle \\
&\quad = \frac{1}{2} \sum_{i=1}^{p} \sum_{s=1}^{p} \langle X_i^* - X_i(1), \mathcal{A}_{si}^*(R_i(1)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(1)) \rangle \\
&\quad = \frac{1}{2} \sum_{i=1}^{p} \sum_{s=1}^{p} [\langle X_i^* - X_i(1), \mathcal{A}_{si}^*(R_i(1)) \rangle \\
&\qquad + \langle X_i^* - X_i(1), \mathcal{U}\mathcal{A}_{si}^*(R_s(1)) \rangle] \\
&\quad = \frac{1}{2} \sum_{i=1}^{p} \sum_{s=1}^{p} \langle \mathcal{A}_{si}(X_i^* - X_i(1)) + \mathcal{A}_{si}\mathcal{U}(X_i^* - X_i(1)), R_s(1) \rangle \\
&\quad = \sum_{i=1}^{p} \sum_{s=1}^{p} \langle \mathcal{A}_{si}(X_i^* - X_i(1)), R_s(1) \rangle \\
&\quad = \sum_{s=1}^{p} \left\langle M_s - \sum_{i=1}^{p} \mathcal{A}_{si}(X_i(1)), R_s(1) \right\rangle \\
&\quad = \sum_{s=1}^{p} \langle R_s(1), R_s(1) \rangle = \sum_{s=1}^{p} \|R_s(1)\|^2 = \|R(1)\|^2.
\end{aligned}$$

For $k = 2$, by algorithm 3.1 and lemma 3.1, we have

$$\begin{aligned}
&\sum_{i=1}^{p}\langle X_i^* - X_i(2), P_i(2)\rangle \\
&\quad = \sum_{i=1}^{p}\left\langle X_i^* - X_i(2), \frac{1}{2}\sum_{s=1}^{p}[\mathcal{A}_{si}^*(R_s(2)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(2))] + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2}P_i(1)\right\rangle \\
&\quad = \sum_{i=1}^{p}\left\langle X_i^* - X_i(2), \frac{1}{2}\sum_{s=1}^{p}[\mathcal{A}_{si}^*(R_s(2)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(2))]\right\rangle \\
&\qquad + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2}\sum_{s=1}^{p}\langle X_i^* - X_{\mathrm{i}}(2), P_i(1)\rangle \\
&\quad = \frac{1}{2}\sum_{i=1}^{p}\sum_{s=1}^{p}\langle X_i^* - X_i(2), \mathcal{A}_{si}^*(R_s(2)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(2))\rangle \\
&\qquad + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2}\sum_{i=1}^{p}\left\langle X_i^* - X_i(1) - \frac{\|R(1)\|_F^2}{\sum_{t=1}^{p}\|P_t(1)\|_F^2}P_i(1), P_i(1)\right\rangle \\
&\quad = \frac{1}{2}\sum_{i=1}^{p}\sum_{s=1}^{p}\langle X_i^* - X_i(2), \mathcal{A}_{si}^*(R_s^{(2)}) + \mathcal{U}\mathcal{A}_{si}^*(R_s^{(2)})\rangle \\
&\qquad + \frac{\|R(2)\|_F^2}{\|R(1)\|_F^2}\sum_{i=1}^{p}\langle X_i^* - X_i(1), P_i(1)\rangle \\
&\qquad - \frac{\|R(2)\|_F^2}{\sum_{t=1}^{p}\|P_t(1)\|_F^2}\sum_{i=1}^{p}\langle P_i(1), P_i(1)\rangle \\
&\quad = \sum_{i=1}^{p}\sum_{s=1}^{p}\langle \mathcal{A}_{si}(X_i^* - X_i(2)), R_s(2)\rangle \\
&\quad = \sum_{s=1}^{p}\left\langle M_s - \sum_{i=1}^{p}\mathcal{A}_{si}(X_i(2)), R_s(2)\right\rangle \\
&\quad = \sum_{s=1}^{p}\|R_s(2)\|_F^2 = \|R(2)\|_F^2.
\end{aligned}$$

Assumed that the conclusion holds for $k = m (m > 2)$, *i.e.* $\sum_{i=1}^{p}\langle X_i^* - X_i(m), P_i(m)\rangle = \|R(m)\|^2$, so for $k = m + 1$, we have

$$
\begin{aligned}
&\sum_{i=1}^{p}\langle X_i^* - X_i(m+1), P_i(m+1)\rangle \\
&\quad = \sum_{i=1}^{p}\left\langle X_i^* - X_i(m+1), \frac{1}{2}\sum_{s=1}^{p}\left[\mathcal{A}_{si}^*(R_s(m+1))\right.\right. \\
&\qquad \left.\left. + \mathcal{U}\mathcal{A}_{si}^*(R_s(m+1))\right] + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2}P_i(m)\right\rangle \\
&\quad = \sum_{i=1}^{p}\left\langle X_i^* - X_i(m+1), \frac{1}{2}\sum_{s=1}^{p}[\mathcal{A}_{si}^*(R_s(m+1)) + \mathcal{U}\right. \\
&\qquad \left.\times \mathcal{A}_{si}^*(R_s(m+1))]\right\rangle + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2}\sum_{i=1}^{p}\langle X_i^* - X_i(m+1), P_i(m)\rangle \\
&\quad = \frac{1}{2}\sum_{i=1}^{p}\sum_{s=1}^{p}\langle X_i^* - X_i(m+1), \mathcal{A}_{si}^*(R_s(m+1)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(m+1))\rangle \\
&\qquad + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2}\sum_{i=1}^{p}\left\langle X_i^* - X_i(m) - \frac{\|R(m)\|_F^2}{\sum_{t=1}^{p}\|P_t(m)\|_F^2}P_i(m), P_i(m)\right\rangle \\
&\quad = \frac{1}{2}\sum_{i=1}^{p}\sum_{s=1}^{p}\langle X_i^* - X_i(m+1), \mathcal{A}_{si}^*(R_s^{(m+1)}) + \mathcal{U}\mathcal{A}_{si}^*(R_s^{(m+1)})\rangle \\
&\qquad + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2}\sum_{i=1}^{p}\langle X_i^* - X_i(m), P_i(m)\rangle \\
&\qquad - \frac{\|R(m+1)\|_F^2}{\sum_{t=1}^{p}\|P_t(m)\|_F^2}\sum_{i=1}^{p}\langle P_i(m), P_i(m)\rangle \\
&\quad = \sum_{i=1}^{p}\sum_{s=1}^{p}\langle \mathcal{A}_{si}(X_i^* - X_i(m+1)), R_s(m+1)\rangle \\
&\quad = \sum_{s=1}^{p}\left\langle M_s - \sum_{i=1}^{p}\mathcal{A}_{si}(X_i(m+1)), R_s(m+1)\right\rangle \\
&\quad = \sum_{s=1}^{p}\|R_s(m+1)\|_F^2 = \|R(m+1)\|_F^2.
\end{aligned}
$$

Thus, we complete the proof by the principle of induction. □

Remark 1. From lemma 3.2, we can conclude that $\|R(k)\|^2 \neq 0$ implies that $\sum_{i=1}^{p} P_i(k) \neq 0$ if problem 3.1 is consistent. Else if there exists a positive number m such that $\|R(m)\|^2 \neq 0$ but $\sum_{i=1}^{p} P_i(m) = 0$, then problem 3.1 must be inconsistent. Hence, the solvability of problem 3.1 can be determined by algorithm 3.1 in the absence of roundoff errors.

Lemma 3.3. For the sequences $\{R_i(k)\}$, $\{P_i(k)\}$ and $\{Z_i(k)\}$, $i = 1, 2, \ldots, p$, generated by algorithm 3.1, satisfy the following equality

$$\sum_{i=1}^{p}\langle R_i(m+1),R_i(n)\rangle=\sum_{i=1}^{p}\langle R_i(m),R_i(n)\rangle-\frac{\|R(m)\|_F^2}{\sum_{t=1}^{p}\|P_t(m)\|_F^2}\sum_{i=1}^{p}\langle P_i(m),Z_i(n)\rangle. \tag{3.1.2}$$

Proof. According to algorithm 3.1, lemma 3.1 and the relation $\mathcal{U}^*=\mathcal{U}$, we have

$$\begin{aligned}
&\sum_{i=1}^{p}\langle R_i(m+1),R_i(n)\rangle\\
&\quad=\sum_{i=1}^{p}\langle R_i(m)-\frac{\|R(m)\|_F^2}{\sum_{t=1}^{p}\|P_t(m)\|_F^2}\sum_{t=1}^{p}\mathcal{A}_{it}(P_t(m)),R_i(n)\rangle\\
&\quad=\sum_{i=1}^{p}\langle R_i(m),R_i(n)\rangle-\frac{\|R(m)\|_F^2}{\sum_{t=1}^{p}\|P_t(m)\|_F^2}\sum_{i=1}^{p}\sum_{t=1}^{p}\langle\mathcal{A}_{it}(P_t(m)),R_i(n)\rangle\\
&\quad=\sum_{i=1}^{p}\langle R_i(m),R_i(n)\rangle-\frac{\|R(m)\|_F^2}{\sum_{t=1}^{p}\|P_t(m)\|_F^2}\sum_{i=1}^{p}\sum_{t=1}^{p}\langle P_t(m),\mathcal{A}_{it}^*(R_i(n))\rangle\\
&\quad=\sum_{i=1}^{p}\langle R_i(m),R_i(n)\rangle\\
&\qquad-\frac{1}{2}\frac{\|R(m)\|_F^2}{\sum_{t=1}^{p}\|P_t(m)\|_F^2}\sum_{i=1}^{p}\sum_{t=1}^{p}\langle P_t(m)+\mathcal{U}P_t(m),\mathcal{A}_{it}^*(R_i(n))\rangle\\
&\quad=\sum_{i=1}^{p}\langle R_i(m),R_i(n)\rangle-\frac{\|R(m)\|_F^2}{\sum_{t=1}^{p}\|P_t(m)\|_F^2}\sum_{t=1}^{p}\langle P_t(m),\\
&\qquad\frac{1}{2}\sum_{i=1}^{p}[\mathcal{A}_{it}^*(R_i(n))+\mathcal{U}\mathcal{A}_{it}^*(R_i(n))\rangle\\
&\quad=\sum_{i=1}^{p}\langle R_i(m),R_i(n)\rangle-\frac{\|R(m)\|_F^2}{\sum_{t=1}^{p}\|P_t(m)\|_F^2}\sum_{t=1}^{p}\langle P_t(m),Z_t(n)\rangle\\
&\quad=\sum_{i=1}^{p}\langle R_i(m),R_i(n)\rangle-\frac{\|R(m)\|_F^2}{\sum_{t=1}^{p}\|P_t(m)\|_F^2}\sum_{i=1}^{p}\langle P_i(m),Z_i(n)\rangle.
\end{aligned}$$

This completes the proof. □

Lemma 3.4. Supposed that problem 3.1 is consistent. If there exists a positive integer m such that $\|R(m)\|^2\neq 0$ for all $m=1,2,\ldots,k$, then the sequences $\{X_i(m)\},\{R_i(m)\}$ and $\{P_i(m)\}$ generated by algorithm 3.1 satisfy the following relations

$$\sum_{i=1}^{p}\langle R_i(m),R_i(n)\rangle=0,\text{ and }\sum_{i=1}^{p}\langle P_i(m),P_i(n)\rangle=0, \tag{3.1.3}$$

in which $m,n=1,2,\ldots,k,m\neq n$.

Proof. For arbitrary two matrices A and B, by using the property of inner product we have $\langle A,B\rangle=\langle B,A\rangle$. Thus, we only need to prove the conclusion holds for all $0\leq n<m\leq k$. In the case of $n=1$, by lemma 3.3 we have

$$
\begin{aligned}
&\sum_{i=1}^{p}\langle R_i(2), R_i(1)\rangle \\
&\quad= \sum_{i=1}^{p}\langle R_i(1), R_i(1)\rangle - \frac{\|R(1)\|^2}{\sum_{t=1}^{r}\|P_t(1)\|^2}\sum_{i=1}^{p}\langle P_i(1), Z_i(1)\rangle \\
&\quad= \sum_{i=1}^{p}\|R_i(1)\|^2 - \frac{\|R(1)\|^2}{\sum_{t=1}^{p}\|P_t(1)\|^2}\sum_{i=1}^{p}\langle P_i(1), P_i(1)\rangle = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{p}\langle P_i(2), P_i(1)\rangle \\
&= \sum_{i=1}^{p}\Big\langle \frac{1}{2}\sum_{s=1}^{p}[\mathcal{A}_{si}^*(R_s(2)) + \mathcal{U}\mathcal{A}_{si}^*(R_s(2)) + \frac{\|R(2)\|^2}{\|R(1)\|^2}P_i(1), P_i(1)\Big\rangle \\
&= \sum_{i=1}^{p}\Big\langle R_s(2), \frac{1}{2}\sum_{s=1}^{p}[\mathcal{A}_{si}(P_i(1)) + \mathcal{U}\mathcal{A}_{si}(P_i(1))]\Big\rangle + \sum_{i=1}^{p}\Big\langle \frac{\|R(2)\|^2}{\|R(1)\|^2}P_i(1), P_i(1)\Big\rangle \\
&= \frac{1}{2}\sum_{s=1}^{p}\sum_{i=1}^{p}\Big\langle R_s(1) - \frac{\|R(1)\|^2}{\sum_{t=1}^{p}\|P_t(1)\|^2}\sum_{i=1}^{p}\mathcal{A}_{si}(P_i(1)), \mathcal{A}_{si}(P_i(1)) \\
&\quad + \mathcal{U}\mathcal{A}_{si}(P_i(1))\Big\rangle + \frac{\|R(2)\|^2}{\|R(1)\|^2}\sum_{i=1}^{p}\|P_i(1)\|^2 \\
&= \frac{1}{2}\sum_{s=1}^{p}\sum_{i=1}^{p}\Big\langle R_s(1), \mathcal{A}_{si}(P_i(1)) + \mathcal{U}\mathcal{A}_{si}(P_i(1))\Big\rangle \\
&\quad - \frac{1}{2}\frac{\|R(1)\|^2}{\sum_{t=1}^{p}\|P_t(1)\|^2}\sum_{i=1}^{p}\sum_{s=1}^{p}\Big\langle \sum_{i=1}^{p}\mathcal{A}_{si}(P_i(1)), \mathcal{A}_{si}(P_i(1)) \\
&\quad + \mathcal{U}\mathcal{A}_{si}(P_i(1))\Big\rangle + \frac{\|R(2)\|^2}{\|R(1)\|^2}\sum_{i=1}^{p}\|P_i(1)\|^2 \\
&= \sum_{s=1}^{p}\Big\langle R_s(1), \sum_{i=1}^{p}\mathcal{A}_{si}(P_i(1))\Big\rangle + \frac{\|R(2)\|^2}{\|R(1)\|^2}\sum_{i=1}^{p}\|P_i(1)\|^2 \\
&\quad - \frac{\|R(1)\|^2}{\sum_{t=1}^{p}\|P_t(1)\|^2}\sum_{s=1}^{p}\sum_{i=1}^{p}\Big\langle \mathcal{A}_{si}(P_i(1)), \sum_{i=1}^{p}\mathcal{A}_{si}(P_i(1))\Big\rangle \\
&= \sum_{s=1}^{p}\Big\langle R_s(1), \frac{\sum_{t=1}^{p}\|P_t(1)\|^2}{\|R(1)\|^2}(R_s(1) - R_s(2))\Big\rangle \\
&\quad - \frac{\|R(1)\|^2}{\sum_{t=1}^{p}\|P_t(1)\|^2}\sum_{s=1}^{p}\Big\langle \frac{\sum_{t=1}^{p}\|P_t(1)\|^2}{\|R(1)\|^2}(R_s(1) - R_s(2)), \\
&\quad \frac{\sum_{t=1}^{p}\|P_t(1)\|^2}{\|R(1)\|^2}(R_s(1) - R_s(2)\Big\rangle + \frac{\|R(2)\|^2}{\|R(1)\|^2}\sum_{i=1}^{p}\|P_i(1)\|^2 \\
&= \frac{\sum_{t=1}^{p}\|P_t(1)\|^2}{\|R(1)\|^2}\sum_{s=1}^{p}\Big\langle R_s(1), R_s(1) - R_s(2)\Big\rangle + \frac{\|R(2)\|^2}{\|R(1)\|^2}\sum_{i=1}^{p}\|P_i(1)\|^2 \\
&\quad - \frac{\sum_{t=1}^{p}\|P_t(1)\|^2}{\|R(1)\|^2}\sum_{s=1}^{p}\Big\langle R_s(1) - R_s(2), R_s(1) - R_s(2)\Big\rangle \\
&= \Big\|\sum_{t=1}^{p}P_t(1)\Big\|^2 - \frac{\sum_{t=1}^{p}\|P_t(1)\|^2}{\|R(1)\|^2}(\|R(1)\|^2 + \|R(2)\|^2) + \frac{\|R(2)\|^2}{\|R(1)\|^2}\sum_{i=1}^{p}\|P_i(1)\|^2 = 0.
\end{aligned}
$$

Now if we assume that $\sum_{i=1}^{p}\langle R_i(m),R_i(n)\rangle=0$ and $\sum_{i=1}^{p}\langle P_i(m),P_i(n)\rangle=0$ hold for all $0\le n<m, 0<m\le k$, next we will prove that $\sum_{i=1}^{p}\langle R_i(m+1),R_i(n)\rangle=0$ and $\sum_{i=1}^{p}\langle P_i(m+1),P_i(n)\rangle=0$ hold for all $0\le n<m+1, 0<m+1\le k$. By the hypothesis and lemma 3.3, in the case of $0\le n<m$, we can obtain

$$\begin{aligned}
&\sum_{i=1}^{p}\langle R_i(m+1),R_i(n)\rangle\\
&\quad=\sum_{i=1}^{p}\left\langle R_i(m),R_i(n)\rangle-\frac{\|R(m)\|^2}{\sum_{t=1}^{p}\|P_t(m)\|^2}\langle P_i(m),Z_i(n)\right\rangle\\
&\quad=-\frac{\|R(m)\|^2}{\sum_{t=1}^{p}\|P_t(m)\|^2}\sum_{i=1}^{p}\left\langle P_i(m),P_i(n)-\frac{\|R(n)\|^2}{\|R(n-1)\|^2}P_i(n-1)\right\rangle\\
&\quad=-\frac{\|R(m)\|^2}{\sum_{t=1}^{p}\|P_t(m)\|^2}\sum_{i=1}^{p}\left\langle P_i(m),P_i(n)\right\rangle\\
&\qquad+\frac{\|R(m)\|^2}{\sum_{t=1}^{p}\|P_t(m)\|^2}\frac{\|R(n)\|^2}{\|R(n-1)\|^2}\sum_{i=1}^{p}\langle P_i(m),P_i(n-1)\rangle=0,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{p}\left\langle P_i(m+1),P_i(n)\right\rangle\\
&\quad=\sum_{i=1}^{p}\left\langle Z_i(m+1)+\frac{\|R(m+1)\|^2}{\|R(m)\|^2}P_i(m),P_i(n)\right\rangle\\
&\quad=\sum_{i=1}^{p}\left\langle Z_i(m+1),P_i(n)\right\rangle+\frac{\|R(m+1)\|^2}{\|R(m)\|^2}\sum_{i=1}^{p}\left\langle P_i(m),P_i(n)\right\rangle\\
&\quad=\sum_{i=1}^{p}\left\langle Z_i(m+1),P_i(n)\right\rangle\\
&\quad=\frac{\sum_{i=1}^{p}\langle R_i(m+1),R_i(n)\rangle-\sum_{i=1}^{p}\langle R_i(m+1),R_i(n+1)\rangle}{\frac{\|R(m)\|^2}{\sum_{t=1}^{p}\|P_t(m)\|^2}}=0.
\end{aligned}$$

In the case of $n=m$, we also have

$$\begin{aligned}
&\sum_{i=1}^{p}\langle R_i(m+1),R_i(m)\rangle\\
&\quad=\sum_{i=1}^{p}\langle R_i(m),R_i(m)\rangle-\frac{\|R(m)\|^2}{\sum_{t=1}^{p}\|P_t(m)\|^2}\sum_{i=1}^{p}\langle P_i(m),Z_i(m)\rangle\\
&\quad=\sum_{i=1}^{p}\|R_i(m)\|^2-\frac{\|R(m)\|^2}{\sum_{t=1}^{p}\|P_t(m)\|^2}\sum_{i=1}^{p}\left\langle P_i(m),P_i(m)-\frac{\|R(m)\|^2}{\|R(m-1)\|^2}P_i(m-1)\right\rangle\\
&\quad=\|R(m)\|^2-\|R(m)\|^2+\frac{\|R(m)\|^2}{\sum_{t=1}^{p}\|P_t(m)\|^2}\frac{\|R(m)\|^2}{\|R(m-1)\|^2}\sum_{i=1}^{p}\left\langle P_i(m),P_i(m-1)\right\rangle\\
&\quad=0,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{p}\langle P_i(m+1), P_i(m)\rangle \\
&\quad= \sum_{i=1}^{p}\left\langle Z_i(m+1) + \frac{\|R(m+1)\|^2}{\|R(m)\|^2}P_i(m), P_i(m)\right\rangle \\
&\quad= \sum_{i=1}^{p}\langle Z_i(m+1), P_i(m)\rangle + \frac{\|R(m+1)\|^2}{\|R(m)\|^2}\sum_{i=1}^{p}\langle P_i(m), P_i(m)\rangle \\
&\quad= \frac{\sum_{i=1}^{p}\langle R_i(m+1), R_i(m)\rangle - \sum_{i=1}^{p}\langle R_i(m+1), R_i(m+1)\rangle}{\frac{\|R(m)\|^2}{\sum_{t=1}^{p}\|P_t(m)\|^2}} \\
&\qquad+ \frac{\|R(m+1)\|^2}{\|R(m)\|^2}\sum_{i=1}^{p}\|P_i(m)\|^2 = 0.
\end{aligned}$$

Therefore, by the principle of induction, the proof has been completed. □

Remark 2. Lemma 3.4 implies that when problem 3.1 is consistent, the sequences $P_i(1), P_i(2), \ldots$, $i = 1, 2, \ldots, p$, are orthogonal to each other in the finite dimension matrix space $\mathcal{S}$. Hence, there exists a positive integer $t \leq mn$ such that$P_i(t) = 0, i = 1, 2, \ldots, p$. According to lemma 3.2, we have$\|R(t)\|^2 = 0$. Thus, it can be concluded that the iteration will be terminated in finite steps in the absence of roundoff errors.

In the following, we will solve the least norm solution of problem 3.1. The following lemmas are needed for our derivation.

Lemma 3.5. [28]Supposed that the consistent system of linear equations $Ax = b$ has a solution $x^* \in R(A^H)$, then x^* is the unique least norm solution of the system of linear equations.

Lemma 3.6. For $\mathcal{A} \in LC^{m\times n, p\times q}$, there exists a unique matrix $M \in \mathfrak{C}^{pq\times mn}$, such that vec $(\mathcal{A}(X)) = M\text{vec}(X)$ for all $X \in \mathfrak{C}^{m\times n}$.

According to lemma 3.6 and the definition of the conjugate operator, one can easily obtain the following corollary.

Corollary 3.1. Let $\mathcal{A}$ and M be the same as those in lemma 3.4, and $\mathcal{A}^*$ be the conjugate operator of $\mathcal{A}$, then vec $(\mathcal{A}^*(Y)) = M^H\text{vec}(Y)$ for all $Y \in \mathfrak{C}^{p\times q}$.

Theorem 3.1. When problem 3.1 is consistent, if the initial matrices are chosen as $X_j(1) = \frac{1}{2}\sum_{i=1}^{p}[\mathcal{A}_{ij}^*(H_j) + \mathcal{U}\mathcal{A}_{ij}^*(H_j)], j = 1, 2, \ldots, p$, in which H_j is an arbitrary matrix, or more especially $X_j(1) = 0, j = 1, 2, \ldots, p$, then the solution group $[X_1^*, X_2^*, \ldots, X_p^*]$ generated by algorithm 3.1 is the least Frobenius norm constraint solution group of the coupled operator matrix equations (3.0.1).

Proof. Now we consider the following coupled operator matrix equations

$$\begin{cases} \sum\limits_{i=1}^{p} \mathcal{A}_{1i}(X_i) = E_1, \\ \sum\limits_{i=1}^{p} \mathcal{A}_{2i}(X_i) = E_2, \\ \qquad\qquad \vdots \\ \sum\limits_{i=1}^{p} \mathcal{A}_{pi}(X_i) = E_p, \\ \sum\limits_{i=1}^{p} \mathcal{A}_{1i}(\mathcal{U}(X_i)) = E_1, \\ \sum\limits_{i=1}^{p} \mathcal{A}_{2i}(\mathcal{U}(X_i)) = E_2, \\ \qquad\qquad \vdots \\ \sum\limits_{i=1}^{p} \mathcal{A}_{pi}(\mathcal{U}(X_i)) = E_p. \end{cases} \tag{3.1.4}$$

If problem 3.1 has a solution $[X_1, X_2, \ldots, X_p]$, then $[X_1, X_2, \ldots, X_p]$ must be a solution of the coupled operator matrix equations (3.1.4). Conversely, if the coupled operator matrix equations (3.1.4) have a solution $[X_1, X_2, \ldots, X_p]$, let $\tilde{X}_j = \frac{X_j + \mathcal{U}(X_j)}{2}, j = 1, 2, \ldots, p$, then it is easy to verify that $[\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_p]$ is a solution to problem 3.1. Therefore, the solvability of problem 3.1 is equivalent to the coupled operator matrix equations (3.1.4).

If we choose the initial matrix $X_j(1) = \frac{1}{2}\sum_{i=1}^{p}[\mathcal{A}_{ij}^*(H_j) + \mathcal{U}\mathcal{A}_{ij}^*(H_j)], j = 1, 2, \ldots, p$, where H_j is an arbitrary matrix in $C^{m_j \times n_j}$, by algorithm 3.1 and remark 2, we can get the solution $[X_1^*, X_2^*, \ldots, X_p^*]$ of problem 3.1 within finite iteration steps. The obtained solution can be stated as $X_j^* = \frac{1}{2}\sum_{i=1}^{p}[\mathcal{A}_{ij}^*(Y_j) + \mathcal{U}\mathcal{A}_{ij}^*(Y_j)], j = 1, 2, \ldots, p$. Since the solution set of problem 3.1 is a subset of that of the coupled operator matrix equations (3.1.4), next we will prove that $[X_1^*, X_2^*, \ldots, X_p^*]$ is the least norm constraint solutions group of the coupled matrix equations (3.1.4). It implies that $[X_1^*, X_2^*, \ldots, X_p^*]$ is also the least norm constraint solutions group of problem 3.1.

Let M_s and N be the matrices such that

$$\mathrm{vec}\Big(\sum_{i=1}^{p} \mathcal{A}_{si}(X_i)\Big) = M_s \begin{pmatrix} \mathrm{vec}(X_1) \\ \vdots \\ \mathrm{vec}(X_p) \end{pmatrix},$$

and

$$\begin{pmatrix} \mathrm{vec}(\mathcal{U}(X_1)) \\ \vdots \\ \mathrm{vec}(\mathcal{U}(X_p)) \end{pmatrix} = N \begin{pmatrix} \mathrm{vec}(X_1) \\ \vdots \\ \mathrm{vec}(X_p) \end{pmatrix},$$

for all $X_j \in \mathfrak{C}^{m_j \times n_j}, j = 1, 2, \ldots, p$. Then the coupled matrix equations (3.1.4) are equivalent to the following coupled matrix equations

$$\begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_p \\ M_1N \\ M_2N \\ \vdots \\ M_pN \end{pmatrix} \begin{pmatrix} \mathrm{vec}(X_1) \\ \vdots \\ \mathrm{vec}(X_p) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}(E_1) \\ \mathrm{vec}(E_2) \\ \vdots \\ \mathrm{vec}(E_p) \\ \mathrm{vec}(E_1) \\ \mathrm{vec}(E_2) \\ \vdots \\ \mathrm{vec}(E_p) \end{pmatrix}.$$

Note that

$$\begin{aligned} \begin{pmatrix} \mathrm{vec}(X_1^*) \\ \vdots \\ \mathrm{vec}(X_p^*) \end{pmatrix} &= \frac{1}{2}\sum_{s=1}^{p}\left[M_s^H \begin{pmatrix} \mathrm{vec}(Y_1) \\ \vdots \\ \mathrm{vec}(Y_p) \end{pmatrix} + N^H M_s^H \begin{pmatrix} \mathrm{vec}(Y_1) \\ \vdots \\ \mathrm{vec}(Y_p) \end{pmatrix} \right] \\ &\in R(M_1^H, M_2^H, \dots, M_p^H, (M_1N)^H, (M_2N)^H, \dots, (M_pN)^H). \end{aligned}$$

Then it follows from lemma 3.5 that $[X_1^*, \dots, X_p^*]$ is the least norm constraint solutions group of the coupled operator matrix equations (3.1.4). This solution is also the least norm constraint solutions of problem 3.1.

3.2 Iterative Method for Solving Problem 3.2

In this section, we consider iterative method for solving nearness problem 3.2 of solution to the matrix equations. Let S denote the solution set of problem 3.1. For given matrix $\widehat{X}_j \in \mathfrak{C}^{m_j \times n_j}, j = 1, 2, \dots, p,$ and arbitrary matrix $X_j \in S$, $j = 1, 2, \dots, p$, we have

$$\begin{aligned} \left\| X_j - \widehat{X}_j \right\|^2 &= \left\| X_j - \frac{\widehat{X}_j + \mathcal{U}(\widehat{X}_j)}{2} - \frac{\widehat{X}_j - \mathcal{U}(\widehat{X}_j)}{2} \right\|^2 \\ &= \left\| X_j - \frac{\widehat{X}_j + \mathcal{U}(\widehat{X})_j}{2} \right\|^2 + \left\| \frac{\widehat{X}_j - \mathcal{U}(\widehat{X})_j}{2} \right\|^2 \\ &\quad - 2\left\langle X_j - \frac{\widehat{X}_j + \mathcal{U}(\widehat{X}_j)}{2}, \frac{\widehat{X}_j - \mathcal{U}(\widehat{X}_j)}{2} \right\rangle. \end{aligned} \tag{3.2.1}$$

Note that

$$\begin{aligned} &\left\langle X_j - \frac{\widehat{X}_j + \mathcal{U}(\widehat{X}_j)}{2}, \frac{\widehat{X}_j - \mathcal{U}(\widehat{X}_j)}{2} \right\rangle \\ &= \left\langle \mathcal{U}(X_j - \frac{\widehat{X}_j + \mathcal{U}(\widehat{X}_j)}{2}), \frac{\widehat{X}_j - \mathcal{U}(\widehat{X}_j)}{2} \right\rangle \\ &= \left\langle X_j - \frac{\widehat{X}_j + \mathcal{U}(\widehat{X}_j)}{2}, \frac{\mathcal{U}(\widehat{X}_j) - \widehat{X}_j}{2} \right\rangle \\ &= -\left\langle X_j - \frac{\widehat{X}_j + \mathcal{U}(\widehat{X}_j)}{2}, \frac{\widehat{X}_j - \mathcal{U}(\widehat{X}_j)}{2} \right\rangle. \end{aligned}$$

This implies $\left\langle X_j - \frac{\widehat{X}_j+\mathcal{U}(\widehat{X}_j)}{2}, \frac{\widehat{X}_j-\mathcal{U}(\widehat{X}_j)}{2} \right\rangle = 0$. Therefore, the relation (3.2.1) can be reduced into

$$\|X_j - \widehat{X}_j\|^2 = \left\|X_j - \frac{\widehat{X}_j+\mathcal{U}(\widehat{X}_j)}{2}\right\|^2 + \left\|\frac{\widehat{X}_j-\mathcal{U}(\widehat{X}_j)}{2}\right\|^2.$$

Hence, $\min_{X_j\in S}\|X_j - \widehat{X}_j\|$ is equivalent to $\min_{X_j\in S}\left\|X_j - \frac{\widehat{X}_j+\mathcal{U}(\widehat{X}_j)}{2}\right\|$, where $X_j - \frac{\widehat{X}_j+\mathcal{U}(\widehat{X}_j)}{2} \in \mathcal{S}$.

Now if we denote $\widetilde{X}_j = X_j - \frac{\widehat{X}_j+\mathcal{U}(\widehat{X}_j)}{2}$ and $\widetilde{E}_s = E_s - \sum_{i=1}^{p}\mathcal{A}_{si}\left(\frac{\widehat{X}_i+\mathcal{U}(\widehat{X}_i)}{2}\right)$, $s = 1, 2, \ldots, p$, then $[\widetilde{X}_1, \ldots, \widetilde{X}_p]$ is the constraint solutions group of the coupled operator matrix equations

$$\Big[\sum_{i=1}^{p}\mathcal{A}_{1i}(\tilde{X}_i), \sum_{i=1}^{p}\mathcal{A}_{2i}(\tilde{X}_i), \ldots, \sum_{i=1}^{p}\mathcal{A}_{pi}(\tilde{X}_i)\Big] = \Big[\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_p\Big], \tilde{X}_i \in \mathcal{S},$$

for $i = 1, 2, \ldots, p$. Thus, problem 3.2 is equivalent to finding the least norm solutions of above coupled operator matrix equations.

Based on the analysis above, it can be summarized that once the unique least norm solutions $[\widetilde{X}_1^*, \ldots, \widetilde{X}_p^*]$ of above coupled operator matrix equations are obtained by applying algorithm 3.1, the unique solutions $[\overline{X}_1, \ldots, \overline{X}_p]$ of problem 3.2 can also be obtained, where $\overline{X}_j = \widetilde{X}_j^* + \frac{\widehat{X}_j+\mathcal{U}(\widehat{X}_j)}{2}, j = 1, 2, \ldots, p$.

3.3 Numerical Example

In this section, we will give three numerical examples to illustrate the efficiency of algorithm 3.1. All the tests are performed by MATLAB 7.0 with machine precision around 10^{-16}.

Example 3.1. Find the least norm symmetric solution of the following matrix equations

$$\begin{cases} A_{11}X_1B_{11} + A_{12}X_2B_{12} = M_1, \\ A_{21}X_1B_{21} + A_{22}X_2B_{22} = M_2, \end{cases} \tag{3.3.1}$$

where $A_{11} = \text{rand}(5,5)$, $B_{11} = \text{rand}(5,5)$, $A_{12} = \text{rand}(5,5)$, $B_{12} = \text{rand}(5,5)$, $A_{21} = \text{rand}(5,5)$, $B_{21} = \text{rand}(5,5)$, $A_{22} = \text{rand}(5,5)$, $B_{22} = \text{rand}(5,5)$.

If we choose $X = \text{rand}(5,5)$, $Y = \text{randi}([1,4],[5,5])$, then we have

$$X_1 = \text{tril}(X,-1) + \text{triu}(X^T,0), \quad X_2 = \text{tril}(Y,-1) + \text{triu}(Y^T,0),$$

$$M_1 = A_{11}X_1B_{11} + A_{12}X_2B_{12}, \quad M_2 = A_{21}X_1B_{21} + A_{22}X_2B_{22}.$$

It can be verified that the generalized coupled matrix equation (3.3.1) are consistent and have unique symmetric solution pair (X_1, X_2). In fact, this solution pair is also the unique minimal norm least squares solution of matrix equation (3.3.1).

If we let the initial matrix $X1 = (X_1(1), X_2(1)) = (\text{zeros}(5,5), \text{zeros}(5,5))$, $X1 = (X_1(1),\ \ X_2(1)) = (\text{ones}(5,5), \text{ones}(5,5))$ and $X1 = (X_1(1), X_2(1)) = (I_5, I_5)$, by applying of algorithm 3.1 we can obtain the symmetric solutions. $r(k)$ and $e(k)$ can be found in figures 3.3.1 and 3.3.2, where

$$e_k = \frac{\|(X_1(k), X_2(k) - (X_1^*, X_2^*)\|}{\|(X_1^*, X_2^*)\|}, \quad r_k = \|R(k)\|.$$

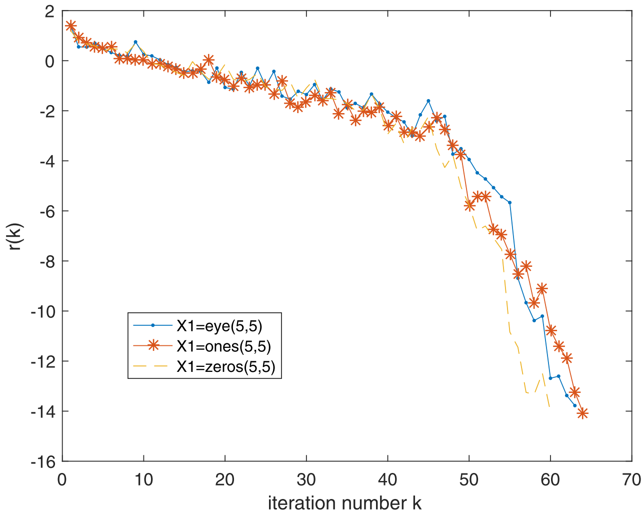

Fig. 3.3.1 – The residual for example 3.1.

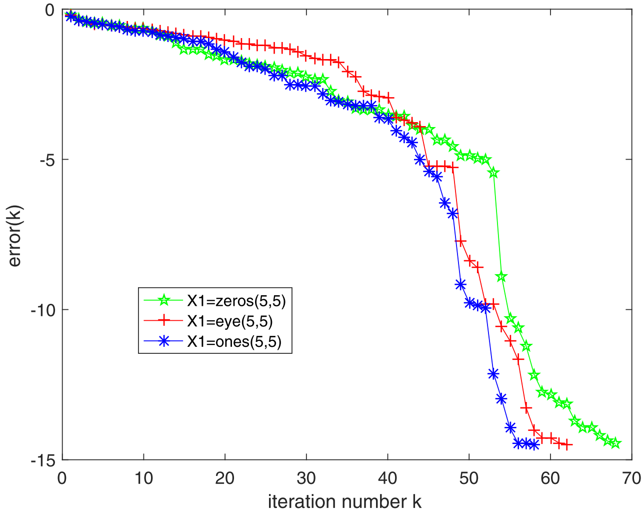

Fig. 3.3.2 – The relative error of solution for example 3.1.

Now if we let $Y_1 = \mathrm{rand}(5,5)$, $Y_2 = \mathrm{randi}([2,3],[5,5])$, then we have

$$\widehat{X}_1 = \mathrm{tril}(Y_1,-1) + \mathrm{triu}(Y_1^T,0), \quad \widehat{X}_2 = \mathrm{tril}(Y_2,-1) + \mathrm{triu}(Y_2^T,0).$$

With the initial matrix pair $\overline{X1} = (\overline{X}_1(1), \overline{X}_2(1)) = (\mathrm{zeros}(5,5), \mathrm{zeros}(5,5))$, $\overline{X1} = (\overline{X}_1(1), \overline{X}_2(1)) = (\mathrm{ones}(5,5), \mathrm{ones}(5,5))$ and $\overline{X1} = (\overline{X}_1(1), \overline{X}_2(1)) = (I_5, I_5)$, by algorithm 3.1 we have the least Frobenius norm solutions $(\overline{X_1}^*, \overline{X_2}^*)$ of the generalized coupled Sylvester matrix equations (3.3.1). The obtained results are presented in figures 3.3.3 and 3.3.4. Therefore, the solutions of the coupled matrix equation (3.3.1) nearness problem are

$$\widetilde{X}_1 = \overline{X_1}^* + \widehat{X_1},$$

and

$$\widetilde{X}_2 = \overline{X_2}^* + \widehat{X_2}.$$

Remark 3. From figures 3.3.1 and 3.3.2, we can see that the residual and the relative error of the solutions to coupled matrix equations (3.3.1) are reduced with the addition of iterative numbers for the different initial matrices $X1 = (X_1(1), X_2(1)) = (\mathrm{zeros}(5,5), \mathrm{zeros}(5,5))$, $X1 = (X_1(1), X_2(1)) = (\mathrm{ones}(5,5), \mathrm{ones}(5,5))$ and $X1 = (X_1(1), X_2(1)) = (I_5, I_5)$, respectively. It showed that the iterative solution smoothly converges to the exact solution. Also, it is showed the presented algorithm in this chapter can be converged for any initial matrices with random coefficient matrices. It is proved that our proposed iterative algorithm is efficient and is a good algorithm.

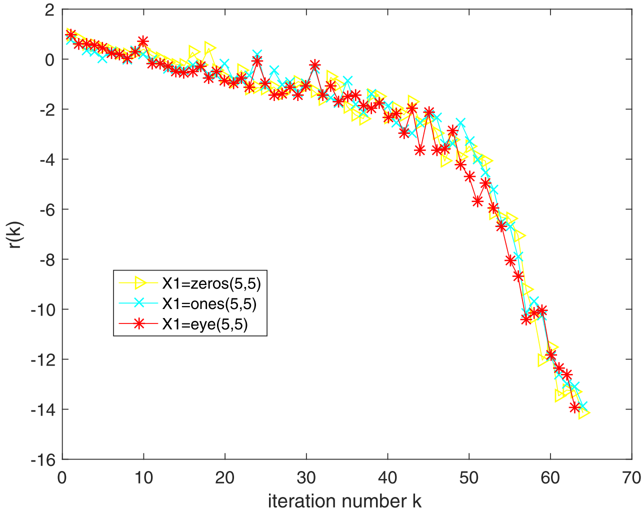

FIG. 3.3.3 – The residual of least norm solution for example 3.1.

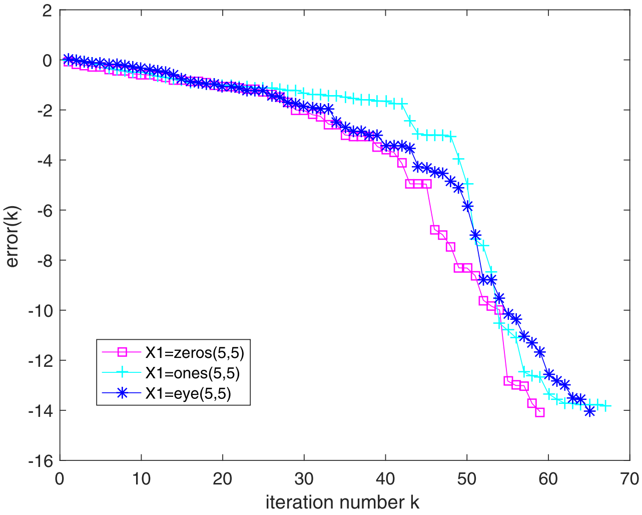

FIG. 3.3.4 – The relative error of least norm solution for example 3.1.

Remark 4. From figures 3.3.3 and 3.3.4, it is shown that the least norm solutions can be derived by using algorithm 3.1 for giving any initial matrices. Hence, if a group of matrices is given, we can obtain the optimal approximation solutions. Therefore, the presented algorithm can solve the nearness problem of coupled matrix equations with random coefficient matrices.

Example 3.2. Find the least norm symmetric solution of the following matrix equations

$$\begin{cases} A_{11}X_1B_{11} + C_{11}X_1^TD_{11} + A_{12}X_2B_{12} + C_{12}X_2^TD_{12} = M_1, \\ A_{21}X_1B_{21} + C_{21}X_{11}^TD_{21} + C_{22}X_{21}^TD_{22} = M_2, \end{cases} \tag{3.3.2}$$

where $A_{11} = \text{tril}(\text{rand}(40,40),1)$, $C_{11} = \text{rand}(40,40)$, $B_{11} = \text{rand}(40,40)$, $D_{11} = \text{rand}\ (40,40)$, $A_{12} = \text{rand}(40,40)$, $C_{12} = \text{rand}(40,40)$, $B_{12} = \text{rand}\ (40,40)$, $D_{12} = \text{rand}\ (40,40)$, $A_{21} = \text{diag}(1+\text{diag}(\text{rand}(40)))$, $C_{21} = \text{diag}(0.5+\text{diag}(\text{rand}(40)))$, $B_{21} = \text{rand}(40,40)$, $D_{21} = \text{rand}(40,40)$, $C_{22} = \text{rand}(40,40)$, $D_{22} = \text{rand}(40,40)$.

If we choose $X = \text{randi}([1,3],[40,40])$, $Y = \text{randi}([2,4],[40,40])$,

$$X_1 = \text{tril}(X,-1) + \text{triu}(X^T,0), \quad X_2 = \text{tril}(Y,-1) + \text{triu}(Y^T,0),$$

then we have

$$M_1 = A_{11}X_1B_{11} + C_{11}X_1^TD_{11} + A_{12}X_2B_{12} + C_{12}X_2^TD_{12},$$
$$M_2 = A_{21}X_1B_{21} + C_{21}X_{11}^TD_{21} + C_{22}X_{21}^TD_{22}.$$

It can be verified that the generalized coupled matrix equations (3.3.2) are consistent and have unique symmetric solution pair (X_1, X_2). Meanwhile, this solution pair is also the unique minimal norm least squares solutions of matrix equations (3.3.2).

If we let the initial matrix $X1 = (X_1(1), X_2(1)) = (I_{40}, I_{40})$, by applying of algorithm 3.1, we can obtain the solutions. $r(k)$ and $e(k)$ can be found in figure 3.3.5. Moreover, if we let the initial matrix $X1 = (X_1(1), X_2(1)) = (2 + 0.1 \times \text{ones}(40, 40), 2 + 0.1 \times \text{ones}(40, 40))$, by applying of algorithm 3.1, we can also obtain the solutions. $r(k)$ and $e(k)$ can be found in figure 3.3.6, where

$$e_k = \frac{\|(X_1(k), X_2(k) - (X_1^*, X_2^*)\|}{\|(X_1^*, X_2^*)\|}, \quad r_k = \|R(k)\|.$$

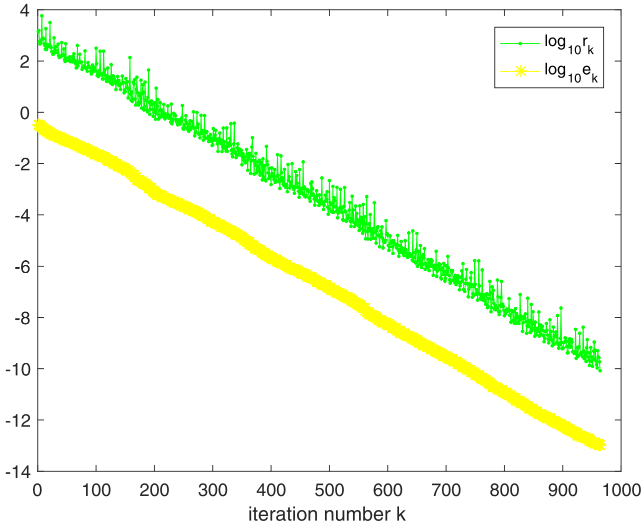

FIG. 3.3.5 – $r(k)$ and $e(k)$ for example 3.2: $(X_1(1), X_2(1)) = (I_{40}, I_{40})$.

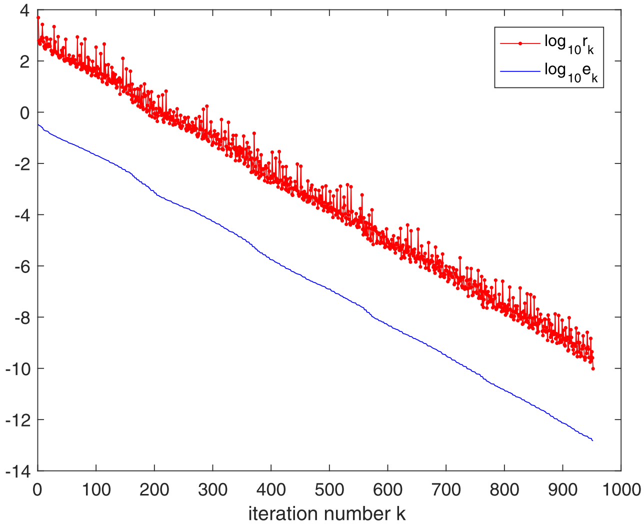

FIG. 3.3.6 – $r(k)$ and $e(k)$ for example 3.2: $(X_1(1), X_2(1)) = (2 + 0.1 \times \text{ones}(40, 40), 2 + 0.1 \times \text{ones}(40, 40))$.

Now if we let $Y_1 = \mathrm{randi}([0,1],[40,40])$, $Y_2 = \mathrm{randi}([2,3],[40,40])$, then we have

$$\widehat{X}_{11} = \mathrm{tril}(Y_1,-1) + \mathrm{triu}(Y_1^T,0), \quad \widehat{X}_{22} = \mathrm{tril}(Y_2,-1) + \mathrm{triu}(Y_2^T,0).$$

With the initial matrix pair $\overline{X1} = (\overline{X}_1(1), \overline{X}_2(1)) = (\mathrm{ones}(40,40), \mathrm{ones}(40,40))$, by algorithm 3.1 we have the least Frobenius norm solutions $(\overline{X_1}^*, \overline{X_2}^*)$ of the generalized coupled Sylvester matrix equation (3.3.2). The obtained results are presented in figure 3.3.7.

Moreover, now if we let $Y_1 = \mathrm{randi}([0,1],[40,40])$, $Y_2 = \mathrm{randi}([0,1],[40,40])$, then we have

$$\widehat{X}_{11} = \mathrm{tril}(Y_1,-1) + \mathrm{triu}(Y_1^T,0), \quad \widehat{X}_{22} = \mathrm{tril}(Y_2,-1) + \mathrm{triu}(Y_2^T,0).$$

with the initial matrix pair $\overline{X1} = (\overline{X}_1(1), \overline{X}_2(1)) = (I_{40}, I_{40})$, we have the least Frobenius norm solutions $(\overline{X_1}^*, \overline{X_2}^*)$ of the generalized coupled Sylvester matrix equation (3.3.2). The obtained results are presented in figure 3.3.8. Therefore, the solutions of the coupled matrix equation (3.3.2) nearness problem are

$$\widetilde{X}_1 = \overline{X_1}^* + \widehat{X_1},$$

and

$$\widetilde{X}_2 = \overline{X_2}^* + \widehat{X_2}.$$

Remark 5. If we provide a group of random coefficient matrices with a bigger dimension, figures 3.3.5 and 3.3.6 show that the presented algorithm is still convergent for any initial matrices. Figures 3.3.7 and 3.3.8 prove that the least norm

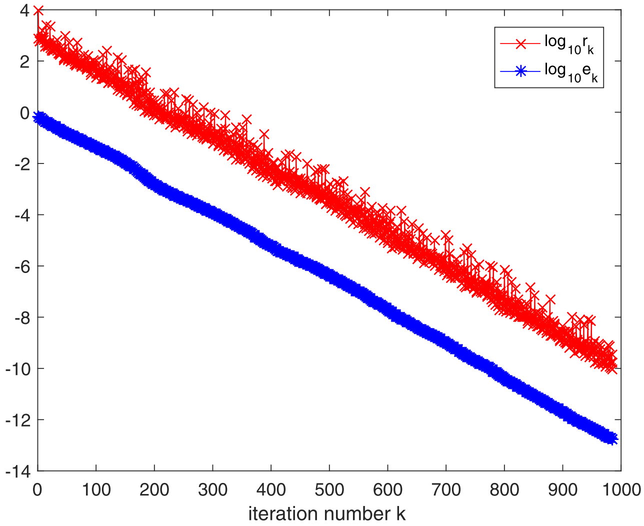

FIG. 3.3.7 – $r(k)$ and $e(k)$ of least Frobenius norm solutions for example 3.2: $(\overline{X}_1(1), \overline{X}_2(1)) = (\mathrm{ones}(40,40), \mathrm{ones}(40,40))$.

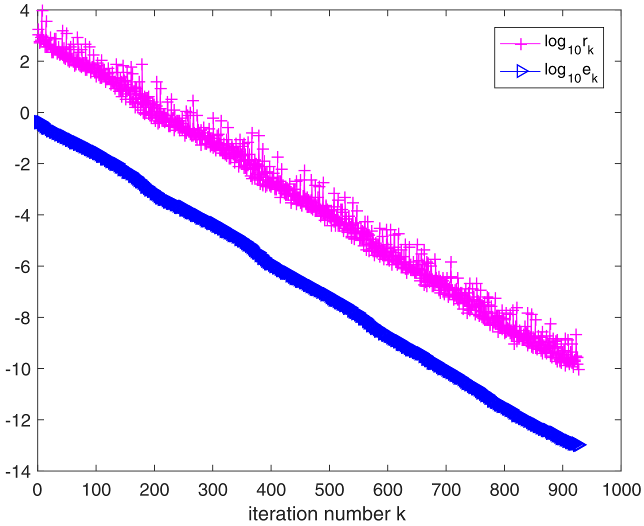

FIG. 3.3.8 – $r(k)$ and $e(k)$ of least Frobenius norm solutions for example 3.2: $\big(\overline{X}_1(1), \overline{X}_2(1)\big) = (I_{40}, I_{40})$.

solutions can be obtained by using algorithm 3.1 for higher dimension random coefficient matrices. Hence, if a group of higher-dimension matrices are given, the optimal approximation solutions can be derived. Therefore, the presented algorithm can solve the nearness problem of coupled matrix equations with higher-dimension random coefficient matrices. This further proves the effectiveness of our proposed iterative algorithm.

Example 3.3. Consider the centrosymmetric solutions of coupled Sylvester-transpose matrix equations

$$\begin{cases} A_{11}XB_{11} + C_{11}X^TD_{11} + A_{12}YB_{12} + C_{12}Y^TD_{12} = E_1, \\ A_{21}XB_{21} + C_{21}X^TD_{21} + C_{22}Y^TD_{22} = E_2, \end{cases} \tag{3.3.3}$$

with the following parametric matrices

$$D_{22} = \begin{pmatrix} -3 & 12 & 11 & 2 & 2 \\ 21 & 3 & 1 & 2 & 3 \\ 1 & 2 & 4 & 1 & 5 \\ 2 & 11 & 3 & 2 & 5 \\ 0 & -1 & -2 & 11 & 4 \end{pmatrix}, E_1 = \begin{pmatrix} 179 & -27 & 58 & 97 & 60 \\ 112 & 177 & 43 & -46 & 164 \\ 63 & 41 & 51 & 269 & 200 \\ -12 & 48 & 2 & 176 & 136 \\ 158 & 94 & 84 & 53 & 75 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 147 & 366 & 332 & 262 & 331 \\ 988 & 212 & 272 & 505 & 587 \\ 388 & 233 & 198 & 149 & 200 \\ 88 & 65 & -57 & -17 & -4 \\ 198 & 109 & 139 & 143 & 179 \end{pmatrix}, B_{12} = \begin{pmatrix} -2 & 1 & 3 & 1 & -1 \\ 4 & 3 & 2 & -1 & 4 \\ 4 & 3 & 2 & 1 & 4 \\ 1 & 2 & 3 & 0 & 5 \\ 2 & 3 & 0 & 3 & 4 \end{pmatrix},$$

$$A_{11} = \begin{pmatrix} 1 & -2 & 1 & -1 & 1 \\ 2 & -2 & 3 & -1 & 0 \\ 2 & 1 & -2 & 2 & -3 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 2 & 0 & 2 & 3 \end{pmatrix}, B_{11} = \begin{pmatrix} 1 & 2 & 1 & -3 & 3 \\ 5 & -2 & 4 & 1 & 0 \\ 1 & -1 & 1 & 4 & 0 \\ 2 & 6 & -2 & 1 & 1 \\ 2 & 1 & 3 & 4 & -6 \end{pmatrix},$$

$$D_{21} = \begin{pmatrix} 1 & -2 & 12 & 3 & 0 \\ 5 & 2 & -4 & 1 & 1 \\ 1 & 1 & -1 & 4 & 3 \\ -2 & 6 & 2 & 1 & 3 \\ 1 & 2 & -1 & 0 & 2 \end{pmatrix}, C_{22} = \begin{pmatrix} 0 & 2 & 0 & 1 & 4 \\ 0 & 11 & 21 & 1 & 5 \\ 4 & 0 & 2 & 1 & 5 \\ 0 & -1 & -1 & 1 & -3 \\ 1 & 2 & 3 & -1 & 2 \end{pmatrix},$$

$$C_{11} = \begin{pmatrix} 0 & 5 & 1 & -1 & 0 \\ 0 & -5 & 3 & 1 & 0 \\ 6 & 1 & 7 & 2 & 3 \\ 2 & 2 & 4 & 1 & 3 \\ 3 & -2 & 2 & 1 & -4 \end{pmatrix}, D_{11} = \begin{pmatrix} -2 & 2 & -1 & 3 & 3 \\ -2 & 2 & -4 & 1 & 3 \\ 1 & 1 & 0 & 0 & 4 \\ 2 & -6 & 2 & 8 & -1 \\ -2 & -1 & -2 & 3 & 4 \end{pmatrix},$$

$$C_{12} = \begin{pmatrix} 3 & 2 & 4 & 1 & 1 \\ 2 & 2 & 0 & 1 & 0 \\ -2 & 1 & 2 & 0 & 0 \\ 1 & 2 & -6 & 1 & 1 \\ 1 & 2 & -1 & -2 & 3 \end{pmatrix}, D_{12} = \begin{pmatrix} 1 & 12 & -1 & -3 & 1 \\ 5 & 2 & 4 & 1 & 1 \\ 10 & 1 & 1 & 4 & 2 \\ -2 & -6 & 2 & 1 & 2 \\ 1 & 2 & 0 & -1 & 0 \end{pmatrix},$$

$$B_{21} = \begin{pmatrix} 4 & -3 & 2 & -2 & 3 \\ 4 & 1 & 2 & -1 & 4 \\ 2 & 3 & 3 & 1 & -2 \\ 4 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 1 & 1 \end{pmatrix}, C_{21} = \begin{pmatrix} 21 & 2 & 11 & 1 & 0 \\ 2 & -2 & -3 & 1 & 0 \\ 2 & -3 & 2 & 2 & 1 \\ 11 & -12 & 5 & 1 & 1 \\ 1 & 2 & 2 & 1 & 4 \end{pmatrix},$$

$$A_{12} = \begin{pmatrix} 1 & -4 & 2 & -6 & 0 \\ 2 & 2 & 6 & 1 & 0 \\ 3 & 2 & 1 & 6 & 1 \\ 2 & 3 & 5 & 2 & -4 \\ 2 & 3 & 0 & 1 & 2 \end{pmatrix}, A_{21} = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 \\ 2 & 2 & 2 & 4 & 2 \\ 1 & 5 & 3 & 1 & 3 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

It can be verified that the generalized coupled matrix equations (3.3.3) are consistent over-generalized centrosymmetric matrices and have the generalized centrosymmetric solution pair (X^*, Y^*) with

$$X^* = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} \in CSR_{Q_1}^{5\times 5},$$

$$Y^* = \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \in CSR_{Q_1}^{5\times 5},$$

where

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in SOR^{5\times 5},$$

$$Q_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \in SOR^{5\times 5}.$$

Taking the initial matrices $(X(1), Y(1)) = (\text{ones}(5,5), \text{ones}(5,5))$ and applying algorithm 3.1, we obtain the solutions

$$X(50) = \begin{pmatrix} 1.0000 & 1.0000 & 0.0000 & 1.9999 & 1.0000 \\ 1.0000 & -0.9999 & 0.0000 & 1.9999 & 0.9999 \\ 0.0000 & 0.0000 & 0.9999 & 0.0000 & 0.0000 \\ 1.9999 & 1.9999 & 0.0000 & 0.0000 & 0.9999 \\ 1.0000 & 0.9999 & 0.0000 & 0.9999 & 0.9999 \end{pmatrix},$$

$$Y(50) = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 2.0000 & 0.0000 \\ 0.0000 & -0.9999 & 1.9999 & 0.0000 & 1.0000 \\ 0.0000 & 1.9999 & 1.0000 & 0.0000 & 0.9999 \\ 2.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.9999 & 0.0000 & 0.9999 \end{pmatrix}.$$

with corresponding residual

$$\|R(50)\| = 9.1981 \times 10^{-12}.$$

The obtained results are presented in figure 3.3.9, where

$$e_k = \frac{\|(X(k), Y(k) - (X^*, Y^*)\|}{\|(X^*, X^*)\|}, \quad r_k = \|R(k)\|.$$

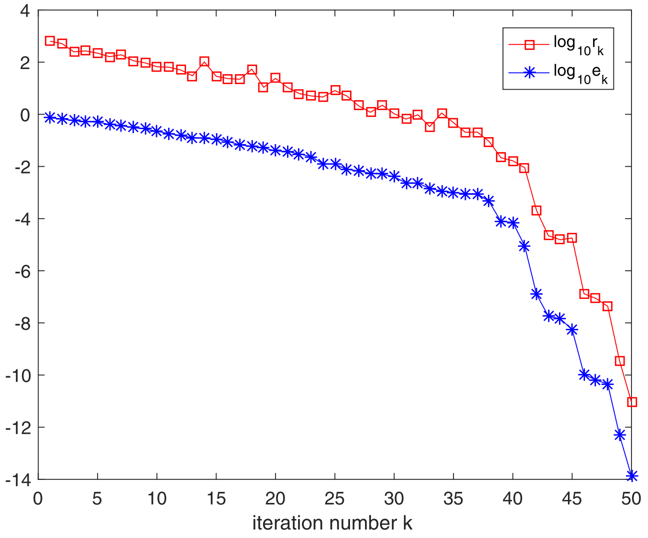

FIG. 3.3.9 – The relative error of solution and the residual for example 3.4.

Now let

$$\widehat{X} = \begin{pmatrix} 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 \end{pmatrix}, \widehat{Y} = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 & 2 \end{pmatrix}.$$

Choosing the initial generalized centrosymmetric matrix pair $(\overline{X}(1), \overline{Y}(1)) = (\text{ones}(5,5), \text{ones}(5,5))$, by algorithm 3.1 we have the least Frobenius norm generalized centrosymmetric solutions of the generalized coupled Sylvester matrix equations (3.3.3) with the following form

$$\overline{X}^* = \overline{X}(50) = \begin{pmatrix} -1.0000 & -0.9999 & 0.0000 & 0.0000 & -1.0000 \\ -0.9999 & -2.9999 & 0.0000 & 0.0000 & -1.0000 \\ 0.0000 & 0.0000 & -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -1.9999 & -1.0000 \\ -1.0000 & -1.0000 & 0.0000 & -1.0000 & -1.0000 \end{pmatrix},$$

$$\overline{Y}^* = \overline{Y}(50) \begin{pmatrix} -0.9999 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -2.9999 & 0.0000 & 0.0000 & -0.9999 \\ 0.0000 & 0.0000 & -0.9999 & 0.0000 & -1.0000 \\ 0.0000 & 0.0000 & 0.0000 & -1.9999 & 0.0000 \\ 0.0000 & -0.9999 & -1.0000 & 0.0000 & -0.9999 \end{pmatrix},$$

with corresponding residual

$$\|R(50)\| = 5.1001 \times 10^{-12}.$$

The obtained results are presented in figure 3.3.10. Therefore, the solutions of the coupled Sylvester-transpose matrix equations nearness problem are

$$\widetilde{X} = \overline{X}^* + \widehat{X} = \begin{pmatrix} 1.0000 & 1.0001 & 0.0000 & 2.0000 & 1.0000 \\ 1.0001 & -0.9999 & 0.0000 & 2.0000 & 1.0001 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 2.0000 & 2.0000 & 0.0000 & 0.0001 & 1.0000 \\ 1.0000 & 1.0000 & 0.0000 & 1.0000 & 1.0000 \end{pmatrix},$$

and

$$\widetilde{Y} = \overline{Y}^* + \widehat{Y} = \begin{pmatrix} 1.0001 & 0.0000 & 0.0000 & 2.0000 & 0.0000 \\ 0.0000 & -0.9999 & 2.0000 & 0.0000 & 1.0001 \\ 0.0000 & 2.0000 & 1.0001 & 0.0000 & 1.0000 \\ 2.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0000 \\ 0.0000 & 1.0001 & 1.0000 & 0.0000 & 1.0001 \end{pmatrix}.$$

Remark 6. From figures 3.3.1–3.3.10, it can be concluded that the presented algorithm 3.1 can solve many kinds of constrained solutions such as the symmetric solution, centrosymmetric solution, and reflective solution. Also, it is showed algorithm 3.1 can computer solutions and least norm solutions of coupled matrix

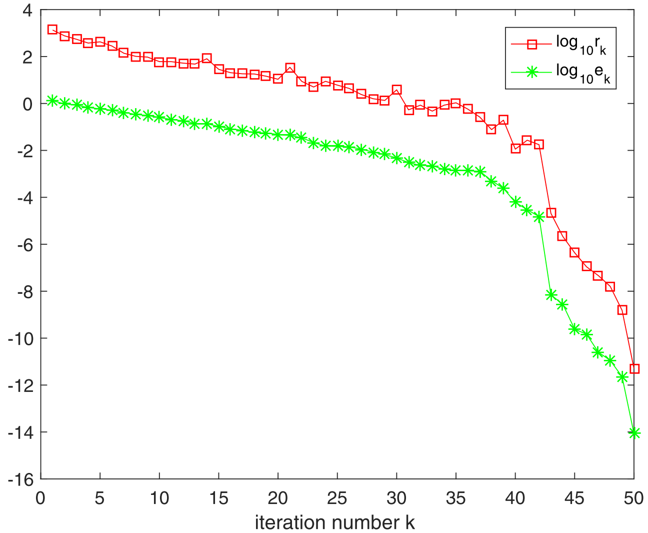

FIG. 3.3.10 – The relative error of the least Frobenius norm generalized centro-symmetric solution and the residual for example 3.4.

equations with some random coefficient matrices, higher dimension random coefficient matrices, and real coefficient matrices for details. Therefore, it can solve the nearness problem of coupled matrix equations, including coupled transpose matrix equations, general coupled matrix equations, and general matrix equations with X and Y unknown. This further proves the effectiveness of our proposed iterative algorithm.

3.4 Control Application

Example 3.4. Consider a linear system which can be described as:

$$\begin{cases} \dot{x} = Ax + Bu + Pw, \\ \dot{w} = Fw, \\ e = Cx + Qw, \end{cases}$$

where $A \in \mathfrak{R}^{n\times n}, B \in \mathfrak{R}^{n\times r}, F \in \mathfrak{R}^{p\times p}, P \in \mathfrak{R}^{n\times p}, C \in \mathfrak{R}^{m\times n}$ and $Q \in \mathfrak{R}^{m\times p}$ are constant matrices, $x \in R^n$, $u \in R^r$ and $e \in R^m$ are the state, the control input and the measurable error output, respectively. The symbol $w \in R^p$ is the exogenous input which includes "reference signals to be tracked" and/or "disturbances to be rejected".

If we assume that (A, B) is controllable, then F is critical stable. If the full information feedback $u = -Kx + Lw$ is applied on the system, then the closed-loop system can result in:

$$\begin{cases} \dot{x} = (A - BK)x + (P + BL)w, \\ \dot{w} = Fw, \\ e = Cx + Qw, \end{cases}$$

The aim of the output regulation problem is to find two matrices K and L such that the matrix $A - BK$ is stable and

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} (Cx(t) + Qw(t)) = 0,$$

which $(x(0), w(0))$ are arbitrary and $(x(0), w(0)) \in R^n \times R^p$. It has been shown in[34] that such a problem is solvable if and only if there exist two matrices X and Y such that

$$AX - XF = BY + R,$$

$$CX + Q = 0.$$

By the theorem 3.1, we can obtain the solution to the matrix equation above.

3.5 Conclusions

In this chapter, in order to solve all kinds of constraint solutions of coupled matrix equations, we construct an iterative algorithm. The presented iterative method can obtain the solutions and least form solutions to the coupled matrix equations with random coefficient matrices, higher random coefficient matrices, and known specific real coefficient matrices. In addition, if a group of matrices is given, we can easily compute the associated optimal approximation solutions. Moreover, the numerical examples show that the solutions to the coupled matrix equations can be obtained for any initial matrices in the absence of roundoff errors. This fully demonstrates the effectiveness of the presented algorithm.

Chapter 4

MCGLS Iterative Algorithm to Linear Conjugate Matrix Equation

In this chapter, we study the generalized Sylvester-conjugate matrix equation

$$\sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j = E, \tag{4.0.1}$$

with known matrices $A_i \in \mathfrak{C}^{p\times m}, B_i \in \mathfrak{C}^{n\times q}, \mathfrak{C}_j \in \mathfrak{C}^{p\times m}, D_j \in \mathfrak{C}^{n\times q}, E \in \mathfrak{C}^{p\times q}$ and unknown matrix $X \in C^{m\times n}$ for $i = 1, 2, \ldots, p, j = 1, 2, \ldots, t$. It is obvious that this class of matrix equation includes various linear matrix equations such as Lyapunov, Sylvester, and the generalized Sylvester matrix equation as special cases. We mainly consider the following two problems.

Problem 4.1. For given $A_i \in \mathfrak{C}^{p\times m}, B_i \in \mathfrak{C}^{n\times q}, C_j \in \mathfrak{C}^{p\times m}, D_j \in \mathfrak{C}^{n\times q}, E \in \mathfrak{C}^{p\times q}$, $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, t$, find $X^* \in \mathfrak{C}^{m\times n}$ such that

$$\begin{aligned} &\left\| \sum_{i=1}^{p} A_i X^* B_i + \sum_{j=1}^{t} C_j \overline{X^*} D_j - E \right\| \\ &= \min_{X\in\mathfrak{C}^{m\times n}} \left\| \sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j - E \right\|. \end{aligned} \tag{4.0.2}$$

Problem 4.2. Let G_r denote the solution set of problem 4.1, for given $X_0 \in \mathfrak{C}^{m\times n}$, find the matrix $\check{X} \in S_r$, such that

$$\|\check{X} - X_0\| = \min_{X\in G_r} \|X - X_0\|. \tag{4.0.3}$$

For X and Y two matrices in $\mathfrak{C}^{m\times n}$, we define the real inner product as $\langle X, Y\rangle = Re[tr(Y^H X)]$. The associated norm is the well-known Frobenius norm. For matrices R, A, B and X with appropriate dimension, a well-known property of the inner product is $\langle R, AXB\rangle = \langle A^H R B^H, X\rangle$. Two matrices X and Y are said to be

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orthogonal if $\langle X, Y\rangle = 0$. First of all, we introduce three lemmas that will be used in the next. And the proof line of lemma 4.2 is similar to the reference[98].

Lemma 4.1. [99]Let U be an inner product space, V be a subspace of U, and $V^{\perp}$ be the orthogonal complement subspace of V. For a given $u \in U$, if there exists a $v_0 \in V$ such that $\|u - v_0\| \leq \|u - v\|$ holds for any $v \in V$, then v_0 is unique and $v_0 \in V$ is the unique minimization vector in V if and only if $(u - v_0) \perp V$, *i.e.* $(u - v_0) \in V^{\perp}$.

Lemma 4.2. Assume that $\hat{R}$ is the residual of equation (4.0.1) corresponding to the matrix $\hat{X} \in \mathfrak{C}^{m\times n}$, that is, $\hat{R} = E - \sum_{i=1}^{p} A_i \hat{X} B_i - \sum_{j=1}^{t} C_j \overline{\hat{X}} D_j$. Then the matrix $\hat{X}$ is the least squares solution of equation (4.0.1) if

$$\sum_{i=1}^{p} A_i^H \hat{R} B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{\hat{R}}\,\overline{D_j}^H = 0. \tag{4.0.4}$$

Proof. We first define the linear subspace $W = \{F | F = \sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j\}$ with $A_i \in \mathfrak{C}^{p\times m}, B_i \in \mathfrak{C}^{n\times q}, C_j \in \mathfrak{C}^{p\times m}, D_j \in \mathfrak{C}^{n\times q}, F \in \mathfrak{C}^{p\times q}$, and $X \in \mathfrak{C}^{m\times n}$, $i = 1, 2, \ldots, p, j = 1, 2, \ldots, t$. Now if we let $\hat{F} = \sum_{i=1}^{p} A_i \hat{X} B_i + \sum_{j=1}^{t} C_j \overline{\hat{X}} D_j$, then $\hat{F} \in W$. Therefore, for any $F \in W$, we have

$$\begin{aligned}
&\langle E - \hat{F}, F\rangle \\
&= \left\langle E - \sum_{i=1}^{p} A_i \hat{X} B_i - \sum_{j=1}^{t} C_j \overline{\hat{X}} D_j, \sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j \right\rangle \\
&= \left\langle \hat{R}, \sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j \right\rangle \\
&= \left\langle \sum_{i=1}^{p} A_i^H \hat{R} B_i^H, X \right\rangle + \left\langle \sum_{j=1}^{t} \overline{C_j}^H \overline{\hat{R}}\,\overline{D_j}^H, X \right\rangle \\
&= \left\langle \sum_{i=1}^{p} A_i^H \hat{R} B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{\hat{R}}\,\overline{D_j}^H, X \right\rangle.
\end{aligned}$$

Hence, if we let $\sum_{i=1}^{p} A_i^H \hat{R} B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{\hat{R}}\,\overline{D_j}^H = 0$, the above equations show that $\langle E - \hat{F}, F\rangle = 0$. It follows from lemma 4.1 that $(E - \hat{F}) \in W^{\perp}$. So the matrix $\hat{X}$ is the least squares solution of equation (4.0.1). □

Lemma 4.3. If we let $\hat{X}$ is a solution of problem 4.1, then any solution $\tilde{X}$ of problem 4.1 can be expressed as $\hat{X} + Z$ where the matrix $Z \in \mathfrak{C}^{m\times n}$ satisfies

$$\sum_{i=1}^{p} A_i Z B_i + \sum_{j=1}^{t} C_j \overline{Z} D_j = 0. \tag{4.0.5}$$

Proof. For any solution $\tilde{X}$ of problem 4.1, if we define the matrix $Z = \tilde{X} - \hat{X}$, then we have $\tilde{X} = \hat{X} + Z$. Now it is showed that equation (4.0.5) holds. By applying lemma 4.2, we can obtain

$$\begin{aligned}
&\left\| \sum_{i=1}^{p} A_i \hat{X} B_i + \sum_{j=1}^{t} C_j \overline{\hat{X}} D_j - E \right\|^2 \\
&= \left\| \sum_{i=1}^{p} A_i \tilde{X} B_i + \sum_{j=1}^{t} C_j \overline{\tilde{X}} D_j - E \right\|^2 \\
&= \left\| \sum_{i=1}^{p} A_i (\hat{X} + Z) B_i + \sum_{j=1}^{t} C_j \overline{(\hat{X} + Z)} D_j - E \right\|^2 \\
&= \left\| \sum_{i=1}^{p} A_i Z B_i + \sum_{j=1}^{t} C_j \overline{Z} D_j - \hat{R} \right\|^2 \\
&= \left\| \sum_{i=1}^{p} A_i Z B_i + \sum_{j=1}^{t} C_j \overline{Z} D_j \right\| + \|\hat{R}\|^2 \\
&\quad - 2\left\langle \sum_{i=1}^{p} A_i Z B_i + \sum_{j=1}^{t} C_j \overline{Z} D_j, \hat{R} \right\rangle \\
&= \left\| \sum_{i=1}^{p} A_i Z B_i + \sum_{j=1}^{t} C_j \overline{Z} D_j \right\| + \|\hat{R}\|^2 \\
&\quad - 2\left\langle Z, \sum_{i=1}^{p} A_i^H \hat{R} B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{\hat{R}} \overline{D_j}^H \right\rangle \\
&= \left\| \sum_{i=1}^{p} A_i Z B_i + \sum_{j=1}^{t} C_j \overline{Z} D_j \right\| + \|\hat{R}\|^2.
\end{aligned}$$

This shows that the equation (4.0.5) holds. □

4.1 MCGLS Iterative Algorithm and Convergence Analysis

The CGLS method is a powerful method for solving the solution of large sparse least squares problem $\min_{x \in \mathfrak{C}^n} \|b - Ax\|$. In this section, we first propose an iterative algorithm by generalizing the CGLS method to solve problem 4.1, then present some basic properties of the algorithm. We also consider finding the least Frobenius norm solution to problem 4.1.

Algorithm 4.1. $x(1) \in R^n$ is an initial matrix, $r(1) = b - Ax(1), s(1) = A^T r(1)$, $p(1) = s(1), \gamma(1) = \|s(1)\|^2$, for $k = 1, 2, 3, \ldots$, repeat the following:

$$\begin{aligned}
&q(k) = Ap(k),\\
&\delta(k) = \gamma(k)/\|q(k)\|^2,\\
&x(k+1) = x(k) + \delta(k)p(k),\\
&r(k+1) = r(k) - \delta(k)q(k),\\
&s(k+1) = A^T r(k+1),\\
&\gamma(k+1) = \|s(k+1)\|^2,\\
&\lambda(k) = \gamma(k+1)/\gamma(k),\\
&p(k+1) = s(k+1) + \lambda(k)p(k).
\end{aligned}$$

Based on the above CGLS method, we propose the following matrix form of the CGLS iterative algorithm (CGLS-M) to solve the least squares problem of the generalized Sylvester-conjugate matrix equation (4.0.1).

Algorithm 4.2

Step 1. Input matrices $A_i \in \mathfrak{C}^{p\times m}, B_i \in \mathfrak{C}^{n\times q}$, $C_j \in \mathfrak{C}^{p\times m}, D_j \in \mathfrak{C}^{n\times q}, E \in \mathfrak{C}^{p\times q}$ and $X(1) \in C^{m\times n}$ for $i = 1, 2, \ldots, p, j = 1, 2, \ldots, t$;
Step 2. Compute

$$\begin{aligned}
&R(1) = E - \sum_{i=1}^{p} A_i X(1) B_i - \sum_{j=1}^{t} C_j \overline{X(1)} D_j;\\
&S(1) = \sum_{i=1}^{p} A_i^H R(1) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{R(1)}\,\overline{D_j}^H;\\
&P(1) = S(1), \quad \gamma(1) = \|S(1)\|^2;
\end{aligned}$$

For $k = 1, 2, 3, \ldots$ repeat the following:

Step 3. If $\|R(k)\| = 0$, then stop and $X(k)$ is the solution of equation (4.0.1), break;
else if $\|R(k)\| \neq 0$ but $\|S(k)\| = 0$, then stop and $X(k)$ is the solution of problem 4.1, break;
else $k := k + 1$;
Step 4. Compute

$$Q(k) = \sum_{i=1}^{p} A_i P(k) B_i + \sum_{j=1}^{t} C_j \overline{P(k)} D_j,$$

$$\begin{aligned}
&\delta(k) = \gamma(k)/\|Q(k)\|^2,\\
&X(k+1) = X(k) + \delta(k)P(k),\\
&R(k+1) = R(k) - \delta(k)Q(k),\\
&S(k+1) = \sum_{i=1}^{p} A_i^H R(k+1) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{R(k+1)}\,\overline{D_j}^H\\
&\qquad\qquad = S(k) - \delta(k)\left(\sum_{i=1}^{p} A_i^H Q(k) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(k)}\,\overline{D_j}^H\right),\\
&\gamma(k+1) = \|S(k+1)\|^2,\\
&\lambda(k) = \gamma(k+1)/\gamma(k),\\
&P(k+1) = S(k+1) + \lambda(k)P(k);
\end{aligned}$$

Step 5. Go to step 3.

Some basic properties of algorithm 4.1 are listed in the following lemmas.

Lemma 4.4. For the sequences $S(k)$, $P(k)$ and $Q(k)$ which are generated by algorithm 4.1, if there exists a positive number r, such that $\|S(u)\| \neq 0$ and $\|Q(u)\| \neq 0, \forall u = 1, 2, \ldots, r$, then the following statements hold for $u, w = 1, 2, \ldots, r$ and $u \neq w$.

(1)$\langle S(u), S(w)\rangle = 0$,
(2)$\langle Q(u), Q(w)\rangle = 0$,
(3)$\langle P(u), S(w)\rangle = 0$.

Proof. Step 1: We prove the conclusion by induction. Because the real inner product is commutative, it is enough to prove three statements for $1 \leq u < w \leq r$.

For $u = 1, w = 2$, by algorithm 4.1, we have

$$\begin{aligned}
&\langle S(1), S(2)\rangle\\
&\quad = \left\langle S(1), S(1) - \delta(1)\left(\sum_{i=1}^{p} A_i^H Q(1) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(1)}\,\overline{D_j}^H\right)\right\rangle\\
&\quad = \|S(1)\|^2 - \delta(1)\left\langle S(1), \sum_{i=1}^{p} A_i^H Q(1) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(1)}\,\overline{D_j}^H\right\rangle\\
&\quad = \|S(1)\|^2 - \delta(1)\left\langle \sum_{i=1}^{p} A_i S(1) B_i + \sum_{j=1}^{t} C_j \overline{S(1)} D_j, Q(1)\right\rangle\\
&\quad = \|S(1)\|^2 - \delta(1)\langle Q(1), Q(1)\rangle = 0,
\end{aligned}$$

$$\begin{aligned}
&\langle Q(1), Q(2)\rangle \\
&\quad = \left\langle Q(1), \sum_{i=1}^{p} A_i P(2) B_i + \sum_{j=1}^{t} C_j \overline{P(2)} D_j) \right\rangle \\
&\quad = \left\langle Q(1), \sum_{i=1}^{p} A_i (S(2) + \lambda(1) P(1)) B_i + \sum_{j=1}^{t} C_j \overline{(S(2) + \lambda(1) P(1))} D_j \right\rangle \\
&\quad = \lambda(1) \| Q(1) \|^2 + \left\langle Q(1), \sum_{i=1}^{p} A_i S(2) B_i \sum_{j=1}^{t} C_j \overline{S(2)} D_j \right\rangle \\
&\quad = \lambda(1) \| Q(1) \|^2 - \frac{1}{\delta(1)} \left\langle \sum_{i=1}^{p} A_i^H (R(2) - R(1)) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{(R(2) - R(1))}\, \overline{D_j}^H, S(2) \right\rangle \\
&\quad = \lambda(1) \| Q(1) \|^2 - \frac{1}{\delta(1)} \langle S(2) - S(1), S(2) \rangle = 0,
\end{aligned}$$

and

$$\langle P(1), S(2)\rangle = \langle S(1), S(2)\rangle = 0.$$

Step 2: In this case, for $u < w < r$ we assume that

$$\langle S(u), S(w)\rangle = 0, \quad \langle Q(u), Q(w)\rangle = 0, \quad \langle P(u), S(w)\rangle = 0.$$

Thus, we can write

$$\begin{aligned}
&\langle S(u), S(w+1)\rangle \\
&\quad = \left\langle S(u), S(w) - \delta(w) \left(\sum_{i=1}^{p} A_i^H Q(w) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)}\, \overline{D_j}^H \right) \right\rangle \\
&\quad = -\delta(w) \left\langle S(u), \sum_{i=1}^{p} A_i^H Q(w) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)}\, \overline{D_j}^H \right\rangle \\
&\quad = -\delta(w) \left\langle \sum_{i=1}^{p} A_i S(u) B_i + \sum_{j=1}^{t} C_j \overline{S(u)} D_j, Q(w) \right\rangle \\
&\quad = -\delta(w) \left\langle \sum_{i=1}^{p} A_i (P(u) - \lambda(u-1) P(u-1)) B_i \right. \\
&\qquad \left. + \sum_{j=1}^{t} C_j \overline{(P(u) - \lambda(u-1) P(u-1))} D_j, Q(w) \right\rangle \\
&\quad = -\delta(w) \langle Q(u) - \lambda(u-1) Q(u-1), Q(w)\rangle = 0,
\end{aligned}$$

$$\begin{aligned}
&\langle P(u), S(w+1)\rangle \\
&\quad = \left\langle P(u), S(w) - \delta(w)\left(\sum_{i=1}^{p} A_i^H Q(w) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H\right)\right\rangle \\
&\quad = -\delta(w)\left\langle P(u), \sum_{i=1}^{p} A_i^H Q(w) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H\right\rangle \\
&\quad = -\delta(w)\left\langle \sum_{i=1}^{p} A_i P(u) B_i + \sum_{j=1}^{t} C_j \overline{P(u)} D_j,\, Q(w)\right\rangle = 0.
\end{aligned}$$

$$\begin{aligned}
&\langle Q(u), Q(w+1)\rangle \\
&\quad = \left\langle Q(u), \sum_{i=1}^{p} A_i P(w+1) B_i + \sum_{j=1}^{t} C_j \overline{P(w+1)} D_j\right\rangle \\
&\quad = \left\langle Q(u), \sum_{i=1}^{p} A_i (S(w+1) + \lambda(w) P(w)) B_i + \sum_{j=1}^{t} C_j \overline{(S(w+1) + \lambda(w) P(w))} D_j\right\rangle \\
&\quad = \left\langle Q(u), \sum_{i=1}^{p} A_i S(w+1) B_i + \sum_{j=1}^{t} C_j \overline{S(w+1)} D_j + \lambda(w) Q(w)\right\rangle \\
&\quad = -\frac{1}{\delta(u)}\left\langle R(u+1) - R(u), \sum_{i=1}^{p} A_i S(w+1) B_i + \sum_{j=1}^{t} C_j \overline{S(w+1)} D_j\right\rangle \\
&\quad = -\frac{1}{\delta(u)}\left\langle \sum_{i=1}^{p} A_i^H (R(u+1) - R(u)) B_i^H \right. \\
&\qquad \left. + \sum_{j=1}^{t} \overline{C_j}^H \overline{(R(u+1) - R(u))}\,\overline{D_j}^H, S(w+1)\right\rangle \\
&\quad = -\frac{1}{\delta(u)}\langle S(u+1) - S(u), S(w+1)\rangle = 0,
\end{aligned}$$

For $u = w$, we have

$$\begin{aligned}
&\langle S(w), S(w+1)\rangle \\
&\quad = \left\langle S(w), S(w) - \delta(w)\left(\sum_{i=1}^{p} A_i^H Q(w) B_i^H \right.\right. \\
&\qquad \left.\left. + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H\right)\right\rangle \\
&\quad = \|S(w)\|^2 - \delta(w)\left\langle S(w), \sum_{i=1}^{p} A_i^H Q(w) B_i^H \right. \\
&\qquad \left. + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|S(w)\|^2 - \delta(w)\left\langle \sum_{i=1}^{p} A_i S(w) B_i + \sum_{j=1}^{t} C_j \overline{S(w)} D_j,\, Q(w) \right\rangle \\
&= \|S(w)\|^2 - \delta(w)\left\langle \sum_{i=1}^{p} A_i (P(w) - \lambda(w-1) P(w-1)) B_i \right. \\
&\quad \left. + \sum_{j=1}^{t} C_j \overline{(P(w) - \lambda(w-1) P(w-1))} D_j,\, Q(w) \right\rangle \\
&= \|S(w)\|^2 - \delta(w)\langle Q(w) - \lambda(w-1) Q(w-1),\, Q(w)\rangle \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
&\langle Q(w),\, Q(w+1)\rangle \\
&\quad = \left\langle Q(w), \sum_{i=1}^{p} A_i P(w+1) B_i + \sum_{j=1}^{t} C_j \overline{P(w+1)} D_j \right\rangle \\
&\quad = \left\langle Q(w), \sum_{i=1}^{p} A_i (S(w+1) + \lambda(w) P(w)) B_i \right. \\
&\qquad \left. + \sum_{j=1}^{t} C_j \overline{(S(w+1) + \lambda(w) P(w))} D_j \right\rangle \\
&\quad = \left\langle Q(w), \sum_{i=1}^{p} A_i S(w+1) B_i + \sum_{j=1}^{t} C_j \overline{S(w+1)} D_j) + \lambda(w) Q(w) \right\rangle \\
&\quad = \lambda(w) \| Q(w)\|^2 - \frac{1}{\delta(w)} \langle R(w+1) - R(w), \\
&\qquad \left. \sum_{i=1}^{p} A_i S(w+1) B_i + \sum_{j=1}^{t} C_j \overline{S(w+1)} D_j \right\rangle \\
&\quad = \lambda(w) \| Q(w)\|^2 - \frac{1}{\delta(w)} \left\langle \sum_{i=1}^{p} A_i^H (R(w+1) - R(w)) B_i^H \right. \\
&\qquad \left. + \sum_{j=1}^{t} \overline{C_j}^H \overline{(R(w+1) - R(w))}\, \overline{D_j}^H,\, S(w+1) \right\rangle \\
&\quad = \lambda(w) \| Q(w)\|^2 - \frac{1}{\delta(w)} \langle S(w+1) - S(w),\, S(w+1)\rangle \\
&\quad = \lambda(w) \| Q(w)\|^2 - \frac{1}{\delta(w)} \|S(w+1)\|^2 \\
&\quad = 0,
\end{aligned}
$$

$$\begin{aligned}
&\langle P(w), S(w+1)\rangle \\
&\quad = \left\langle P(w), S(w) - \delta(w)\left(\sum_{i=1}^{p} A_i^H Q(w) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H\right)\right\rangle \\
&\quad = \langle S(w) + \lambda(w-1)S(w-1) + \lambda(w-1)\lambda(w-2)S(w-2) \\
&\qquad + \cdots + \lambda(w-1)\ldots\lambda(1)S(1), S(w)\rangle - \delta(w)\langle P(w), \\
&\qquad \sum_{i=1}^{p} A_i^H Q(w) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H \Bigg\rangle \\
&\quad = \|S(w)\|^2 - \delta(w)\left\langle \sum_{i=1}^{p} A_i P(w) B_i + \sum_{j=1}^{t} C_j \overline{P(w)} D_j, Q(w)\right\rangle = 0.
\end{aligned}$$

By the principle of induction, we draw a conclusion. □

Lemma 4.5. If there exists a positive number l such that $\delta(l) = 0$ or $\delta(l) = \infty$ in algorithm 4.1, then $X(l)$ is a solution of problem 4.1.

Proof. If $\delta(l) = 0$, we have $\|S(l)\|^2 = 0$. If $\delta(l) = \infty$, we have $\|Q(l)\|^2 = 0$. It is showed that

$$\begin{aligned}
\|S(l)\|^2 &= \langle S(l) + \lambda(l-1)S(l-1) + \lambda(l-1)\lambda(l-2)S(l-2) + \cdots \\
&\quad + \lambda(l-1)\ldots\lambda(1)S(1), S(l)\rangle \\
&= \langle P(l), S(l)\rangle \\
&= \left\langle P(l), \sum_{i=1}^{p} A_i^H R(l) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{R(l)}\,\overline{D_j}^H \right\rangle = 0.
\end{aligned}$$

Therefore, for $\delta(l) = 0$ or $\delta(l) = \infty$ we have

$$S(l) = \sum_{i=1}^{p} A_i^H R(l) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{R(l)}\,\overline{D_j}^H = 0.$$

So by lemma 4.2, we can conclude that $X(l)$ is the solution to problem 4.1. □

Theorem 4.1. If the matrix equation (4.0.1) is consistent, then, for any arbitrary initial matrix $X(l)$, the solution X^* of problem 4.1 can be obtained by using algorithm 4.1 within a finite number of iterations in the absence of roundoff errors.

Proof. By lemma 4.4, the set $S(i), i = 1, 2, 3, \ldots, m \times n$ is an orthogonal basis of the real inner product space $C^{m\times n}$ with dimension $m \times n$. Therefore, we can obtain $\|S(m \times n + 1)\| = 0$. It is also showed that $X(m \times n + 1)$ is the solution of problem 4.1 in the absence of roundoff errors. □

Theorem 4.2. Supposed that the initial matrix is $\sum_{i=1}^{p} A_i^H F(l) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{F(l)}\,\overline{D_j}^H$, where $F(1) \in \mathfrak{C}^{m\times n}$ is an arbitrary matrix, or especially $X(1) = 0$, then the solution X^* generated by algorithm 4.1 is the least norm solution of problem 4.1.

Proof. Let the initial matrix be $X(1)=\sum_{i=1}^{p}A_i^H F(l)B_i^H+\sum_{j=1}^{t}\overline{C_j}^H\overline{F(l)}\overline{D_j}^H$, then, it follows from algorithm 4.1 that the generated matrix $X(k)$ can be easily expressed as $X(k)=\sum_{i=1}^{p}A_i^H F(k)B_i^H\ +\sum_{j=1}^{t}\overline{C_j}^H\overline{F(k)}\overline{D_j}^H$ for certain matrices $F(k)\in\mathfrak{C}^{m\times n}$ for $k=2,3,\ldots$. This implies that there exists a matrix $F^*\in\mathfrak{C}^{m\times n}$ such that $X^*=\sum_{i=1}^{p}A_i^H F^* B_i^H+\sum_{j=1}^{t}\overline{C_j}^H\overline{F}^*\overline{D_j}^H$. Now if we let $\tilde{X}^*$ is an arbitrary solution of problem 4.1, then, by using lemma 4.3, there exists the matrix $Z^*\in\mathfrak{C}^{m\times n}$ such that $\tilde{X}^*=X^*+Z^*$ and $\sum_{i=1}^{p}A_iZ^*B_i+\sum_{j=1}^{t}C_j\overline{Z^*}D_j=0$. Thus, one can obtain

$$\begin{aligned}\langle X^*,Z^*\rangle&=\left\langle\sum_{i=1}^{p}A_i^H F^* B_i^H+\sum_{j=1}^{t}\overline{C_j}^H\overline{F}^*\overline{D_j}^H,Z^*\right\rangle\\&=\left\langle F^*,\sum_{i=1}^{p}A_iZ^*B_i+\sum_{j=1}^{t}C_j\overline{Z^*}D_j\right\rangle=0.\end{aligned}$$

By applying this relation, one can obtain

$$\begin{aligned}\|\tilde{X}^*\|^2&=\|X^*+Z^*\|^2\\&=\|X^*\|^2+\|Z^*\|^2+2\langle X^*,Z^*\rangle=\|X^*\|^2+\|Z^*\|^2\geq\|X^*\|^2.\end{aligned}$$

This shows that the solution X^* is the least Frobenius norm solution of problem 4.1. □

Similar to[98], the minimization property of the proposed algorithm is stated as follows. This property shows that algorithm 4.1 converges smoothly.

Theorem 4.3. For any initial matrix $X(1)\in\mathfrak{C}^{m\times n}$, we have

$$\begin{aligned}&\left\|\sum_{i=1}^{p}A_iX(k+1)B_i+\sum_{j=1}^{t}C_j\overline{X(k+1)}D_j-E\right\|^2\\&=\min_{X\in\psi_k}\left\|\sum_{i=1}^{p}A_iXB_i+\sum_{j=1}^{t}C_j\overline{X}D_j-E\right\|^2,\end{aligned}$$

where $X(k+1)$ is generated by algorithm 4.1 at the $k+1$-th iteration and ψ_k presents an affine subspace which has the following form

$$\psi_k=X(1)+\mathrm{span}(P(1),P(2),\ldots,P(k)).$$

Proof. For any matrix $X\in\psi_k$, it follows from the second formula in theorem 4.3 that there exist numbers $\alpha_1,\alpha_2,\ldots,\alpha_k$ such that

$$X=X(1)+\sum_{l=1}^{k}\alpha_lP(l).$$

Now we define the continuous and differentiable function f with respect to the variable $\alpha_1,\alpha_2,\ldots,\alpha_k$ as

$$\begin{aligned} f(\alpha_1, \alpha_2, \ldots, \alpha_k) = \Bigg\| & \sum_{i=1}^{s} A_i \Big(X(1) + \sum_{l=1}^{k} \alpha_l P(l)\Big) B_i \\ & + \sum_{j=1}^{t} C_j \overline{\Big(X(1) + \sum_{l=1}^{k} \alpha_l P(l)\Big)} D_j - E \Bigg\|^2 . \end{aligned}$$

By lemma 4.4 one can obtain

$$\begin{aligned} & f(\alpha_1, \alpha_2, \ldots, \alpha_k) \\ & = \Bigg\| \sum_{i=1}^{s} A_i X(1) B_i + \sum_{j=1}^{t} C_j \overline{X(1)} D_j - E \\ & \quad + \sum_{l=1}^{k} \alpha_l \left[\sum_{i=1}^{s} A_i P(l) B_i + \sum_{j=1}^{t} C_j \overline{P(l)} D_j \right] \Bigg\|^2 \\ & = \|R(1)\|^2 + \sum_{l=1}^{k} \alpha_l^2 \|Q(1)\|^2 - 2\alpha_l \langle Q(1), R(1) \rangle . \end{aligned}$$

Now we consider the problem of minimizing the function $f(\alpha_1, \alpha_2, \ldots, \alpha_k)$. It is obvious that

$$\min_{\alpha_l} f(\alpha_1, \alpha_2, \ldots, \alpha_k) = \min_{X \in \psi_k} \left\| \sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j - E \right\|^2 .$$

For this function, the minimum occurs when

$$\frac{\partial(\alpha_1, \alpha_2, \ldots, \alpha_k)}{\partial \alpha_l} = 0 \quad \text{for} \quad l = 1, 2, \ldots, k.$$

Thus, we can get

$$\alpha_l = \frac{\langle Q(l), R(l) \rangle}{\|Q(l)\|^2} .$$

From algorithm 4.1, one can obtain

$$R(1) = R(l) + \delta(l-1)\, Q(l-1) + \delta(l-2)\, Q(l-2) + \cdots + \delta(1)\, Q(1).$$

Therefore, it follows from lemma 4.4 that

$$\begin{aligned} \alpha_l & = \frac{\langle Q(l), R(l) \rangle}{\|Q(l)\|^2} = \frac{\langle P(l), \sum_{i=1}^{s} A_i^H R(l) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{R(l)} \overline{D_j}^H \rangle}{\|Q(l)\|^2} \\ & = \frac{\langle P(l), S(l) \rangle}{\|Q(l)\|^2} = \frac{\langle S(l) + \lambda(l-1) P(l-1), S(l) \rangle}{\|Q(l)\|^2} \\ & = \frac{\|S(l)\|^2}{\|Q(l)\|^2} = \delta(l). \end{aligned}$$

Thus, we complete the proof. □

By theorem 4.1, the solution generalized by algorithm 4.1 at the $k+1$-th iteration for any initial matrix minimizes the residual norm in the affine subspace ψ_k. Also one has

$$\left\|\sum_{i=1}^{s} A_i X(k+l) B_i + \sum_{j=1}^{t} C_j \overline{X(k+l)} D_j - E\right\|^2$$
$$\leq \left\|\sum_{i=1}^{s} A_i X(k) B_i + \sum_{j=1}^{t} C_j \overline{X(k)} D_j - E\right\|^2,$$

which shows that the sequence of the norm of residuals $\|R(1)\|, \|R(2)\|, \ldots$ is monotonically decreasing. This decent property of the norm of the residuals shows that algorithm 4.1 processes fast and smooth convergence.

Now we will solve problem 4.2. For a given matrix X_0, by problem 4.1 we can get

$$\min_{X\in\mathfrak{C}^{m\times n}} \left\|\sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j - E\right\|$$
$$= \min_{X\in\mathfrak{C}^{m\times n}} \left\|\sum_{i=1}^{p} A_i (X - X_0) B_i + \sum_{j=1}^{t} C_j \overline{(X - X_0)} D_j\right.$$
$$\left. -\left(E - \sum_{i=1}^{p} A_i X_0 B_i - \sum_{j=1}^{t} C_j \overline{X_0} D_j\right)\right\|.$$

If we let the set $E_1 = E - \sum_{i=1}^{p} A_i X_0 B_i - \sum_{j=1}^{t} C_j \overline{X_0} D_j$ and $X_1 = X - X_0$, then problem 4.2 is equivalent to find the least Frobenius norm solution X_1^* of

$$\min_{X_1\in\mathfrak{C}^{m\times n}} \left\|\sum_{i=1}^{p} A_i X_1 B_i + \sum_{j=1}^{t} C_j \overline{X_1} D_j - E_1\right\|,$$

which can be computed by using algorithm 4.1 with the initial matrix $X_1(1) = \sum_{i=1}^{s} A_i^H F B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{F} \overline{D_j^H}$ where $F \in C^{m\times n}$ is an arbitrary matrix, or especially $X_1(1) = 0$. Thus the solution of problem 4.2 $\|\check{X} - X_0\| = \min_{X\in G_r} \|X - X_0\|$ can be stated as

$$\check{X} = X_1^* + X_0.$$

It's showed that based on the problem 4.1, we can solve the problem 4.2. In other words, we can obtain the solution of problem 4.2 by using the solution of problem 4.1.

4.2 Numerical Example

In this section, we shall give some numerical examples to illustrate the efficiency of algorithm 4.1. All the tests are performed by MATLAB with machine precision around 10^{-16}.

Example 4.1. Find the least Frobenius norm solution of the following generalized Sylvester-conjugate matrix equation

$$AXB + C\overline{X}D = M, \tag{4.2.1}$$

where

$$A = \begin{pmatrix} 1+i & 1-i & 6i & 4+3i \\ 2+i & 0 & 12 & 10 \\ 5+6i & 2+3i & 11-2i & i \\ 1 & 12 & 0 & 9i \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 3-2i & 4+6i & 9+8i \\ 4+6i & 11 & 12 & 9 \\ 0 & 12 & 15 & 18 \\ 2-i & -9i & 12 & 11 \end{pmatrix},$$

$$C = \begin{pmatrix} i & 9 & 15 & 2i \\ 11 & 2+3i & 12 & 11 \\ 18 & 11-9i & 21 & i \\ 9+8i & 11 & i & 8i \end{pmatrix}, D = \begin{pmatrix} 9+2i & -i & i & 6+8i \\ 4i & 19 & 0 & 11 \\ 23 & 26+i & 9i & 9 \\ 23 & 0 & 6 & 8i \end{pmatrix},$$

$$M = \begin{pmatrix} 4660-5011i & 6322-1536i & 2161+1789i & 6035+1940i \\ 14900-1787i & 13144-1058i & 5872+5983i & 12675+8798i \\ 12135-12509i & 12910-5709i & 4671+6089i & 11126+4083i \\ 10031+8092i & 12415+3909i & 3793+9126i & 5328+12279i \end{pmatrix}.$$

It can be verified that the generalized Sylvester-conjugate matrix equation (4.2.1) is consistent and has the solution

$$X = \begin{pmatrix} 4+3i & 2+i & 11 & 6 \\ 9+6i & 11+2i & 8+9i & 9 \\ 9i & 4 & 2i & 11i \\ 23 & 4 & 7 & 12 \end{pmatrix}.$$

If we let the initial matrix $X_1 = 0$, applying algorithm 4.1, we obtain the solution with the corresponding residual

$$r(126) = 9.8915 \times 10^{-16},$$

and relative error

$$e(126) = 1.3293 \times 10^{-14}.$$

The obtained results are presented in figures 4.2.1 and 4.2.2, where

$$r_k = \log_{10}\|E - AX(k)B - C\overline{X(k)}D\|,$$

and

$$\mathrm{err}(k) = \frac{\|X(k) - X\|}{\|X\|}.$$

If we let the initial matrices be $X(1) = 0, X(1) = I_4, X(1) = \mathrm{ones}(4,4)$, respectively, applying algorithm 4.1, the residual and the relative error of the solution are presented in figures 4.2.3 and 4.2.4.

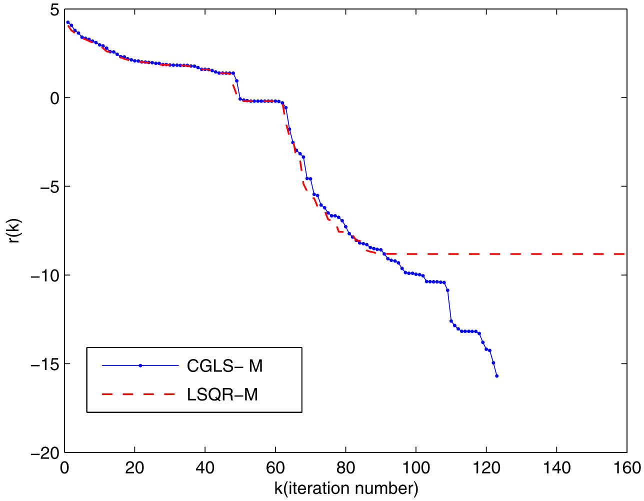

Fig. 4.2.1 – The residual of solution for example 4.1.

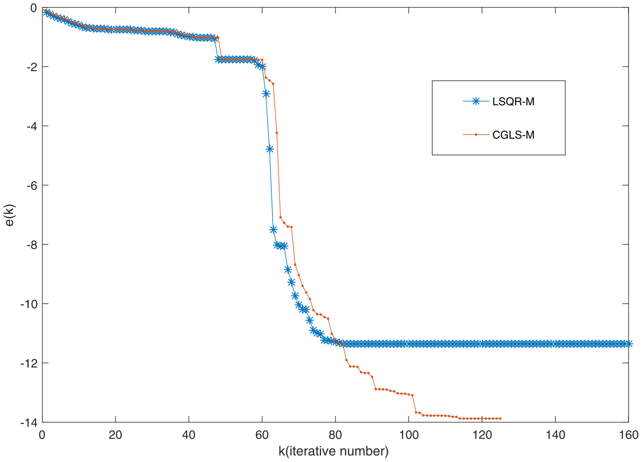

Fig. 4.2.2 – The relative error of solution for example 4.1.

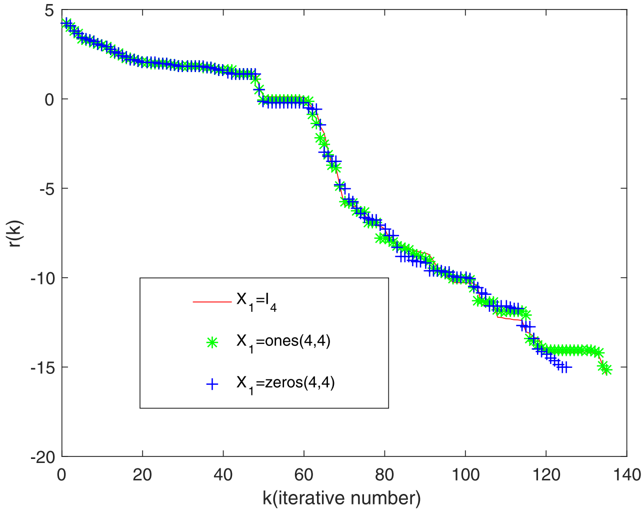

FIG. 4.2.3 – The residual of solution for different initial values for example 4.1.

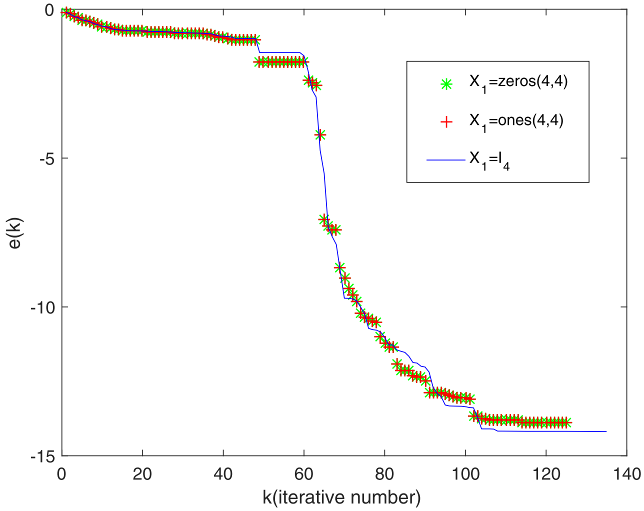

FIG. 4.2.4 – The relative error of solution for different initial values for example 4.1.

Remark 1. From figures 4.2.1 and 4.2.2, the residual and the relative error of the solution to matrix equation (4.2.1) are reduced with the addition of an iterative number. It showed that the iterative solution converges to the exact solution. Moreover, with the addition of an iterative number, the residual and the relative error of the LSQR-M iterative algorithm are not varied. Finally, the residual and the relative error of the iterative algorithm are smaller than those of the LSQR-M iterative algorithm. It is proved that our proposed method (MCGLS iterative method or CGLS-M iterative algorithm) is efficient and is a good algorithm.

Remark 2. From figures 4.2.3 and 4.2.4, for the different initial value $X(1) = 0$, $X(1) = I_4, X(1) = \text{ones}(4,4)$, respectively, the iterative solution to the matrix equation (4.2.1) all converges to the exact solution to the matrix equation (4.2.1).

Example 4.2. Find the least Frobenius norm solution of the following matrix equation

$$AX + \overline{X}D = M, \tag{4.2.2}$$

where

$$D = \begin{pmatrix} 9+2i & -i & i & 6+8i & 12 & 23 \\ 4i & 19 & 0 & 11 & 3i & 3+4i \\ 23 & 26+i & 9i & 9 & 12 & 23 \\ 23 & 0 & 6 & 8i & 2 & 18+9i \\ 112 & 123+i & 119 & 21 & 29+9i & 3i \\ 11 & 2 & 3i & 4+28i & 89+9i & 788 \end{pmatrix},$$

$$A = \begin{pmatrix} 1+2i & 11-3i & 12+6i & 4+23i & 11 & 2 \\ 2+11i & 0 & 12 & 10 & 34+5i & 11 \\ 5+6i & 2+3i & 11-2i & i & 34 & 45 \\ 1 & 12 & 0 & 9i & 13+45i & 3 \\ 23 & 43 & 21 & 99i & 45i & 4+9i \\ 34 & 89 & 45i & 2+3i & 4+7i & 4 \end{pmatrix},$$

$$M = \begin{pmatrix} 1349+1310i & 829-172i & 604+229i & 1547+2029i & 8166+846i & 53917+3054i \\ 2951+2863i & 4222+22i & 4948+311i & 5997+782i & 7743-2175i & 2060-23329i \\ 574+3690i & 4159-959i & 6374-1083i & 5778+166i & 14398-1699i & 10406-3223i \\ -802+1004i & 1885-105i & 2110+827i & 2515+7660i & 10150+6537i & 62506+2197i \\ 14159+2417i & 16561+1277i & 16974+3140i & 2939+6494i & 6763+9222i & 11737+10689i \\ 24695-1540i & 29322+289i & 25160+2003i & 14052-275i & 12010-18641i & 4965-182152i \end{pmatrix}.$$

It can be verified that the generalized Sylvester-conjugate matrix equation (4.2.2) is consistent and has the solution

$$X = \begin{pmatrix} 4+3i & 2+i & 11 & 6 & 3i & 67 \\ 9+6i & 11+2i & 8+9i & 9 & 23 & 34i \\ 9i & 4 & 2i & 11i & 3+10i & 12+18i \\ 23 & 4 & 7 & 12i & 12+3i & 78 \\ 2+78i & 2+9i & 17 & 119 & 127 & 12 \\ 56i & 79 & 118 & 19 & 191 & 237i \end{pmatrix}.$$

If we let the initial matrix $X(1)=0$, applying algorithm 4.1, we obtain the solution, which is

$$X(587) = \left(\begin{matrix} 4.0000+3.0000i & 2.0000+0.99999i & 11.0000 \\ 8.9999+6.0000i & 10.9999+2.0000i & 7.9999+9.0000i \\ 8.9999i & 3.9999 & 1.9999i \\ 22.9999 & 3.9999 & 6.9999 \\ 2.0000+77.9999i & 2.0000+8.9999i & 17.0000 \\ 56.0000i & 79.0000 & 117.9999 \end{matrix}\right.$$

$$\left.\begin{matrix} 6.0000 & 2.9999i & 67.0000 \\ 9.0000 & 23.0000 & 33.9999i \\ 11.0000i & 3.0000+1.0000i & 12.0000+18.0000i \\ 12.0000i & 12.0000+3.0000i & 78.0000 \\ 118.9999 & 126.9999 & 11.9999 \\ 19.0000 & 191.0000 & 237.0000 \end{matrix}\right),$$

with corresponding residual

$$r(587) = 6.9270 \times 10^{-16},$$

and relative error

$$e(587) = 7.9381 \times 10^{-14}.$$

The obtained results are presented in figures 4.2.5 and 4.2.6, where

$$r_k = \log_{10}\|E - AX(k)B - C\overline{X(k)}D\|,$$

and

$$\operatorname{err}(k) = \frac{\|X(k) - X\|}{\|X\|}.$$

If we let the initial matrices be $X(1)=0, X(1)=I_4, X(1)=\text{ones}(4,4)$, respectively, applying algorithm 4.1, the residual and the relative error of solution are presented in figures 4.2.7 and 4.2.8.

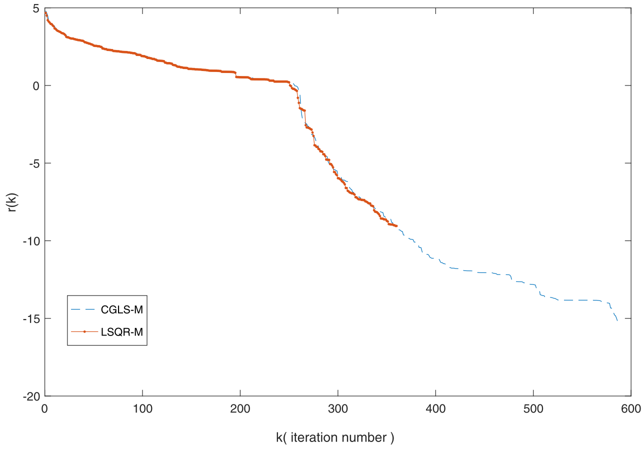

Fig. 4.2.5 – The residual of solution for example 4.2.

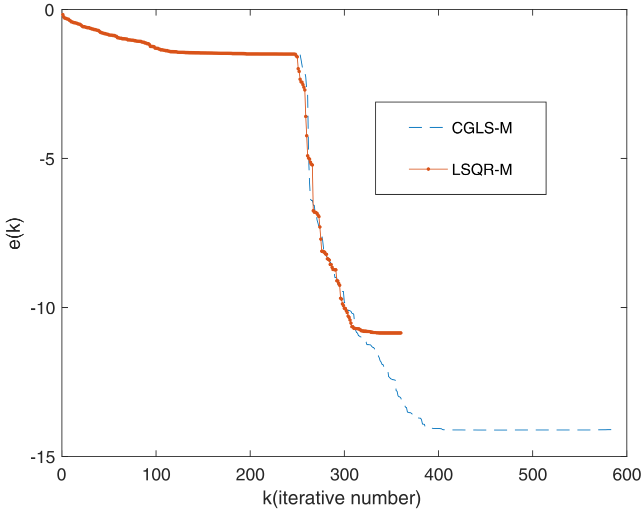

Fig. 4.2.6 – The relative error of solution for example 4.2.

Remark 3. Figures 4.2.5 and 4.2.6, showed that the iterative solution converges to the exact solution. Moreover, with the addition of the iterative number, the residual and the relative error of the iterative algorithm are smaller than those of the LSQR-M iterative algorithm. It is proved that our proposed method (CGLS-M iterative algorithm) is efficient and is a good algorithm. From figures 4.2.7 and 4.2.8, for the different initial value $X(1) = 0, X(1) = I_4, X(1) = \text{ones}(4, 4)$, respectively, the iterative solution to the matrix equation (4.2.2) all converges to its exact solution.

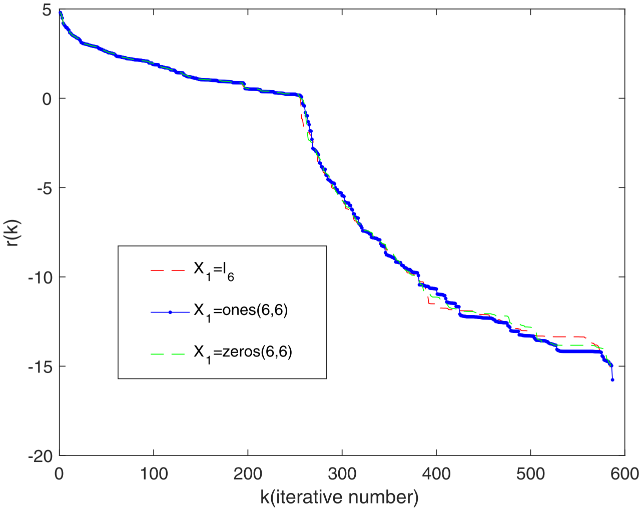

Fig. 4.2.7 – The residual of solution for different initial values for example 4.2.

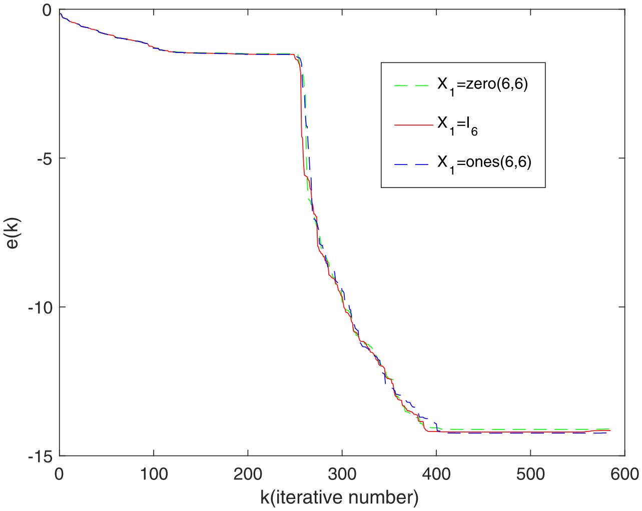

Fig. 4.2.8 – The relative error of solution for different initial values for example 4.2.

4.3 Control Application

In linear systems, the problem of eigenstructure assignment is to find a control law such that the system matrix of the closed-loop system has the desired eigenvalues and multiplicities, and simultaneously determines the corresponding eigenvectors and generalized eigenvectors.

Now we consider the following discrete-time anti-linear system

$$x(t+1) = A\overline{x(t)} + B\overline{u(t)}, \tag{4.3.1}$$

where $x(t) \in \mathfrak{C}^n$ is the state, and $u(t) \in \mathfrak{C}^r$ is the input. $A \in \mathfrak{C}^{n\times n}$ and $C \in \mathfrak{C}^{n\times r}$ are the system matrix and input matrix, respectively. If the state feedback

$$u(t) = Kx(t), \tag{4.3.2}$$

is applied to the system (4.3.1), the closed-loop system is obtained as

$$x(t+1) = (A + B\overline{K})\overline{x(t)}. \tag{4.3.3}$$

Given a prescribed matrix $F \in \mathfrak{C}^{n\times n}$, find a state feedback controller in the form of (4.3.2) such that the system matrix $(A + B\overline{K})$ of the closed-loop system (4.3.3) is consimilar to the matrix F, and simultaneously determine a nonsingular matrix $X \in \mathfrak{C}^{n\times n}$ satisfying

$$X^{-1}(A + B\overline{K})\overline{X} = F. \tag{4.3.4}$$

Due to the non-singularity of X, the equation (4.3.4) can equivalently written as

$$A\overline{X} + B\overline{KX} = XF. \tag{4.3.5}$$

Let $\overline{KX} = W$, then, the equation (4.3.5) can be transformed into the following equation

$$A\overline{X} + BW = XF, \tag{4.3.6}$$

which is the so-called Sylvester conjugate matrix equation. By algorithm 4.1, we can obtain the solution of equation (4.3.6).

4.4 Conclusions

In this chapter, the MCGLS iterative algorithm is constructed to solve the least Frobenius norm solution of the generalized Sylvester-conjugate matrix equation (4.0.1). By this algorithm, for any initial matrix $X(1)$, a solution X^* can be obtained in finite iteration steps in the absence of roundoff errors, and the least Frobenius norm solution can be obtained by choosing any initial matrix. In addition, by use of this iterative method, the optimal approximation solution $\check{X}$ to a given matrix X_0 can be derived by first finding the least Frobenius norm solution of a new corresponding matrix equation. And we have checked the obtained theory by the numerical examples. The given numerical examples show that the proposed iterative algorithms are quite more efficient than the existing LSQR iterative algorithm[90,93,94].

Chapter 5

MCGLS Iterative Algorithm to Linear Conjugate Transpose Matrix Equation

The aim of this chapter is to present an iterative algorithm for solving the generalized Sylvester conjugate transpose matrix equation. There are many problems such as in mathematics, physics, and engineering, leading to solving linear matrix equations. For example, a digital filter can be characterized by state variable equations

$$x(n+1) = Ax(n) + bu(n), \tag{5.0.1}$$

and

$$y(n) = cx(n) + du(n). \tag{5.0.2}$$

The solutions K and W of the Lyapunov matrix equations

$$K = AKA^T + bb^T, \tag{5.0.3}$$

and

$$W = A^T WA + c^T c, \tag{5.0.4}$$

can analyze the quantization noise generated by a digital filter[3]. It can be shown that certain control problem, such as pole/eigenstructure assignment and observe design of (5.0.1), are closely related with the following generalized Sylvester matrix equation

$$\sum_{k=1}^{p} A_k X B_k + \sum_{j=1}^{q} C_j Y D_j = E. \tag{5.0.5}$$

The generalized Sylvester conjugate transpose matrix equation

$$\sum_{i=1}^{r} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j + \sum_{s=1}^{k} G_s X^T H_s + \sum_{l=1}^{u} M_l X^H N_l = E, \tag{5.0.6}$$

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with known matrices $A_i \in \mathfrak{C}^{p\times m}, B_i \in \mathfrak{C}^{n\times q}$, $C_j \in \mathfrak{C}^{p\times m}, D_j \in \mathfrak{C}^{n\times q}$, $G_s \in \mathfrak{C}^{p\times n}$, $H_s \in \mathfrak{C}^{m\times q}, M_l \in \mathfrak{C}^{p\times n}, N_l \in \mathfrak{C}^{m\times q}, E \in \mathfrak{C}^{p\times q}$ and unknown matrix $X \in C^{m\times n}$ for $i = 1, 2, \ldots, r; j = 1, 2, \ldots, t; s = 1, 2, \ldots, k; l = 1, 2, \ldots, u.$, which is discussed in this chapter, is the generalization of the above matrix equation (5.0.5). Obviously, this class of matrix equation includes various linear matrix equations such as Lyapunov, Sylvester and the generalized Sylvester matrix equations as special cases.

This chapter is concerned to find the least squares solution of generalized Sylvester-conjugate transpose matrix equation (5.0.6) by generalizing CGLS iterative algorithm. For details, we mainly consider the following two problems.

Problem 5.1. For given $A_i \in \mathfrak{C}^{p\times m}, B_i \in \mathfrak{C}^{n\times q}$, $C_j \in \mathfrak{C}^{p\times m}, D_j \in \mathfrak{C}^{n\times q}$, $G_s \in \mathfrak{C}^{p\times n}$, $H_s \in \mathfrak{C}^{m\times q}, M_l \in \mathfrak{C}^{p\times n}, N_l \in \mathfrak{C}^{m\times q}, E \in \mathfrak{C}^{p\times q}, i = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, t; s = 1, 2, \ldots, k$ and $l = 1, 2, \ldots, u$. find $X^* \in \mathfrak{C}^{m\times n}$ such that

$$\left\|\sum_{i=1}^{r} A_i X^* B_i + \sum_{j=1}^{t} C_j \overline{X^*} D_j + \sum_{s=1}^{k} G_s {X^*}^T H_s + \sum_{l=1}^{u} M_l {X^*}^H N_l - E\right\| = \min_{X\in C^{m\times n}} \left\|\sum_{i=1}^{r} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j + \sum_{s=1}^{k} G_s X^T H_s + \sum_{l=1}^{u} M_l X^H N_l - E\right\|. \tag{5.0.7}$$

Problem 5.2. Let G_r denote the solution set of problem 5.1, for given $X_0 \in \mathfrak{C}^{m\times n}$, find the matrix $\check{X} \in S_r$, such that

$$\|\check{X} - X_0\| = \min_{X\in G_r} \|X - X_0\|. \tag{5.0.8}$$

Firstly, let I_n and S_n denote the $n \times n$ unit matrix and reverse unit matrix respectively. For X and Y two matrices in $\mathfrak{C}^{m\times n}$, we define the real inner product as $\langle X, Y\rangle = Re[tr(Y^H X)]$. The associated norm is the well-known Frobenius norm. For matrices R, A, B and X with appropriate dimension, a well-known property of the inner product is $\langle R, AXB\rangle = \langle A^H R B^H, X\rangle$. Two matrices X and Y are said to be orthogonal if $\langle X, Y\rangle = 0$. First of all, we present two lemmas. And the proof line of lemma 5.2 is similar to the reference[98].

Lemma 5.1. [99]Let U be an inner product space, V be a subspace of U, and $V^{\perp}$ be the orthogonal complement subspace of V. Given $w \in U$, if there exists a $w_0 \in V$ such that $\|w - v_0\| \leq \|w - v\|$ holds for any $v \in V$, then v_0 is unique and $v_0 \in V$ is the unique minimization vector in V if and only if $(w - v_0) \perp V$, *i.e.* $(w - v_0) \in V^{\perp}$.

Lemma 5.2. Assume that $\hat{R}$ is the residual of equation (5.0.6) corresponding to the matrix $\hat{X} \in \mathfrak{C}^{m\times n}$, that is, $\hat{R} = E - \sum_{i=1}^{r} A_i \hat{X} B_i - \sum_{j=1}^{t} C_j \overline{\hat{X}} D_j - \sum_{s=1}^{k} G_s \hat{X}^T H_s - \sum_{l=1}^{u} M_l \hat{X}^H N_l$. Then the matrix $\hat{X}$ is the least squares solution of equation (5.0.6) if

$$\sum_{i=1}^{r} A_i^H \hat{R} B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{\hat{R}}\,\overline{D_j}^H + \sum_{s=1}^{k} \overline{H_s}\hat{R}^T \overline{G_s} + \sum_{l=1}^{u} N_l \hat{R}^H M_l = 0. \tag{5.0.9}$$

Proof. We first define the linear subspace $W = \{F | F = \sum_{i=1}^{r} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j + \sum_{s=1}^{k} G_s X^T H_s + \sum_{l=1}^{u} M_l X^H N_l\}$ with $A_i \in \mathfrak{C}^{p\times m}, B_i \in \mathfrak{C}^{n\times q}$, $C_j \in \mathfrak{C}^{p\times m}, D_j \in \mathfrak{C}^{n\times q}$, $G_s \in \mathfrak{C}^{p\times n}$, $H_s \in \mathfrak{C}^{m\times q}$, $M_l \in \mathfrak{C}^{p\times n}$, $N_l \in \mathfrak{C}^{m\times q}$, $E \in \mathfrak{C}^{p\times q}$, $F \in \mathfrak{C}^{p\times q}$, and $X \in \mathfrak{C}^{m\times n}$, $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, t$; $s = 1, 2, \ldots, k$ and $l = 1, 2, \ldots, u$. Next, if we let $\hat{F} = \sum_{i=1}^{r} A_i \hat{X} B_i + \sum_{j=1}^{t} C_j \overline{\hat{X}} D_j + \sum_{s=1}^{k} G_s \hat{X}^T H_s + \sum_{l=1}^{u} M_l \hat{X}^H N_l$, then $\hat{F} \in W$ holds. Therefore, for any $F \in W$, one can obtain

$$\begin{aligned}
&\langle E - \hat{F}, F\rangle \\
&= \left\langle E - \sum_{i=1}^{p} A_i \hat{X} B_i - \sum_{j=1}^{t} C_j \overline{\hat{X}} D_j - \sum_{s=1}^{k} G_s \hat{X}^T H_s - \sum_{l=1}^{u} M_l \hat{X}^H N_l, \right. \\
&\qquad \left. \sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j + \sum_{s=1}^{k} G_s X^T H_s + \sum_{l=1}^{u} M_l X^H N_l \right\rangle \\
&= \left\langle \hat{R}, \sum_{i=1}^{r} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j + \sum_{s=1}^{k} G_s X^T H_s + \sum_{l=1}^{u} M_l X^H N_l \right\rangle \\
&= \left\langle \sum_{i=1}^{r} A_i^H \hat{R} B_i^H, X \right\rangle + \left\langle \sum_{j=1}^{t} \overline{C_j}^H \overline{\hat{R}}\,\overline{D_j}^H, X \right\rangle + \left\langle \sum_{s=1}^{k} \overline{H_s}\hat{R}^T \overline{G_s}, X \right\rangle \\
&\qquad + \left\langle \sum_{l=1}^{u} N_l \hat{R}^H M_l, X \right\rangle \\
&= \left\langle \sum_{i=1}^{r} A_i^H \hat{R} B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{\hat{R}}\,\overline{D_j}^H + \sum_{s=1}^{k} \overline{H_s}\hat{R}^T \overline{G_s} + \sum_{l=1}^{u} N_l \hat{R}^H M_l, X \right\rangle.
\end{aligned}$$

In this case, if we suppose $\sum_{i=1}^{r} A_i^H \hat{R} B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{\hat{R}}\,\overline{D_j}^H + \sum_{s=1}^{k} \overline{H_s}\hat{R}^T \overline{G_s} + \sum_{l=1}^{u} N_l \hat{R}^H M_l = 0$, then by the above equations it is apparent that $\langle E - \hat{F}, F\rangle = 0$. By lemma 5.1, we have $(E - \hat{F}) \in W^{\perp}$. Thus, the matrix $\hat{X}$ is the least squares solution of equation (5.0.6). $\square$

Lemma 5.3. If we let $\hat{X}$ be a solution of problem 5.1, then any solution $\tilde{X}$ of problem 5.1 can be expressed as $\hat{X} + Z$ where the matrix $Z \in \mathfrak{C}^{m\times n}$ satisfies

$$\sum_{i=1}^{r} A_i Z B_i + \sum_{j=1}^{t} C_j \overline{Z} D_j + \sum_{s=1}^{k} G_s Z^T H_s + \sum_{l=1}^{u} M_l Z^H N_l = 0. \tag{5.0.10}$$

Proof. For any solution $\tilde{X}$ of problem 5.1, if we define the matrix $Z = \tilde{X} - \hat{X}$, then $\tilde{X} = \hat{X} + Z$ holds. Now it is showed that equation (5.0.10) holds. By applying lemma 5.2, we can obtain

$$\begin{aligned}
&\left\|\sum_{i=1}^{r} A_i\hat{X}B_i + \sum_{j=1}^{t} C_j\overline{\hat{X}}D_j + \sum_{s=1}^{k} G_s\hat{X}^T H_s + \sum_{l=1}^{u} M_l\hat{X}^H N_l - E\right\|^2 \\
&= \left\|\sum_{i=1}^{r} A_i\tilde{X}B_i + \sum_{j=1}^{t} C_j\overline{\tilde{X}}D_j + \sum_{s=1}^{k} G_s\tilde{X}^T H_s + \sum_{l=1}^{u} M_l\tilde{X}^H N_l - E\right\|^2 \\
&= \Bigg\|\sum_{i=1}^{r} A_i(\hat{X}+Z)B_i + \sum_{j=1}^{t} C_j\overline{(\hat{X}+Z)}D_j \\
&\quad + \sum_{s=1}^{k} G_s(\hat{X}+Z)^T H_s + \sum_{l=1}^{u} M_l(\hat{X}+Z)^H N_l - E\Bigg\|^2 \\
&= \left\|\sum_{i=1}^{r} A_iZB_i + \sum_{j=1}^{t} C_j\overline{Z}D_j + \sum_{s=1}^{k} G_sZ^T H_s + \sum_{l=1}^{u} M_lZ^H N_l - \hat{R}\right\|^2 \\
&= \left\|\sum_{i=1}^{r} A_iZB_i + \sum_{j=1}^{t} C_j\overline{Z}D_j + \sum_{s=1}^{k} G_sZ^T H_s + \sum_{l=1}^{u} M_lZ^H N_l\right\|^2 + \left\|\hat{R}\right\|^2 \\
&\quad - 2\left\langle \sum_{i=1}^{r} A_iZB_i + \sum_{j=1}^{t} C_j\overline{Z}D_j + \sum_{s=1}^{k} G_sZ^T H_s + \sum_{l=1}^{u} M_lZ^H N_l, \hat{R}\right\rangle \\
&= \left\|\sum_{i=1}^{r} A_iZB_i + \sum_{j=1}^{t} C_j\overline{Z}D_j + \sum_{s=1}^{k} G_sZ^T H_s + \sum_{l=1}^{u} M_lZ^H N_l\right\|^2 + \left\|\hat{R}\right\|^2 \\
&\quad - 2\left\langle Z, \sum_{i=1}^{r} A_i^H\hat{R}B_i^H + \sum_{j=1}^{t} \overline{C_j}^H\overline{\hat{R}}\,\overline{D_j}^H + \sum_{s=1}^{k} \overline{H_s}\hat{R}^T\overline{G_s} + \sum_{l=1}^{u} N_l\hat{R}^H M_l \right\rangle \\
&= \left\|\sum_{i=1}^{r} A_iZB_i + \sum_{j=1}^{t} C_j\overline{Z}D_j + \sum_{s=1}^{k} G_sZ^T H_s + \sum_{l=1}^{u} M_lZ^H N_l\right\|^2 + \left\|\hat{R}\right\|^2.
\end{aligned}$$

Therefore, the equation (5.0.9) holds. □

5.1 MCGLS Iterative Algorithm and Convergence Analysis

In this section, firstly, in order to solve problem 5.1 an iterative algorithm is proposed, then some basic properties of the algorithm are given. Next finding the least Frobenius norm solution to problem 5.2 is discussed. As is well known, the CGLS method is a powerful method for obtaining the solution of large sparse least squares problem $\min_{x\in\mathbb{C}^n}\|b - Ax\|$. It is stated as follows.

Algorithm CGLS.[29] $x(1) \in \Re^n$ is an initial guess, $r(1) = b - Ax(1)$, $s(1) = A^T r(1), p(1) = s(1), \gamma(1) = \|s(1)\|^2$, for $k = 1, 2, 3, \ldots$ repeat the following:

$$\begin{aligned}
&q(k) = Ap(k),\\
&\delta(k) = \gamma(k)/\|q(k)\|^2,\\
&x(k+1) = x(k) + \delta(k)p(k),\\
&r(k+1) = r(k) - \delta(k)q(k),\\
&s(k+1) = A^T r(k+1),\\
&\gamma(k+1) = \|s(k+1)\|^2,\\
&\lambda(k) = \gamma(k+1)/\gamma(k),\\
&p(k+1) = s(k+1) + \lambda(k)p(k).
\end{aligned}$$

For solving the least squares problem of the generalized Sylvester-conjugate transpose matrix equation (5.0.6), based on the above CGLS method in vector form, the following matrix form of the CGLS iterative algorithm (MCGLS or CGLS-M) is presented.

Algorithm 5.1

Step 1. Input matrices $A_i \in \mathfrak{C}^{p\times m}$, $B_i \in \mathfrak{C}^{n\times q}$, $C_j \in \mathfrak{C}^{p\times m}, D_j \in \mathfrak{C}^{n\times q}$, $G_s \in \mathfrak{C}^{p\times n}$, $H_s \in \mathfrak{C}^{m\times q}$, $M_l \in \mathfrak{C}^{p\times n}$, $N_l \in \mathfrak{C}^{m\times q}$, $E \in \mathfrak{C}^{p\times q}$, $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, t$; $s = 1, 2, \ldots, k$ and $l = 1, 2, \ldots, u$. and $X(1) \in \mathfrak{C}^{m\times n}$;
Step 2. Compute

$$\begin{aligned}
R(1) = E &- \sum_{i=1}^{r} A_i X(1) B_i - \sum_{j=1}^{t} C_j \overline{X(1)} D_j\\
&- \sum_{s=1}^{k} G_s X(1)^T H_s + \sum_{l=1}^{u} M_l X(1)^H N_l;\\
S(1) = &\sum_{i=1}^{p} A_i^H R(1) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{R(1)}\,\overline{D_j}^H\\
&+ \sum_{s=1}^{k} \overline{H_s} R(1)^T \overline{G_s} + \sum_{l=1}^{u} N_l R(1)^H M_l;\\
P(1) = &S(1), \gamma(1) = \|S(1)\|^2;
\end{aligned}$$

For $k = 1, 2, 3, \ldots$ repeat the following:
Step 3. If $\|R(k)\| = 0$, then stop and $X(k)$ is the solution of equation (5.0.6), break;
else if $\|R(k)\| \neq 0$ but $\|S(k)\| = 0$, then stop and $X(k)$ is the solution of problem 5.1, break;
else $k := k + 1$;

Step 4. Compute

$$\begin{aligned}
Q(k) &= \sum_{i=1}^{r} A_i P(k) B_i + \sum_{j=1}^{t} C_j \overline{P(k)} D_j \\
&\quad + \sum_{s=1}^{k} G_s P(k)^T H_s + \sum_{l=1}^{u} M_l P(k)^H N_l; \\
\delta(k) &= \gamma(k) / \| Q(k) \|^2; \\
X(k+1) &= X(k) + \delta(k) P(k); \\
R(k+1) &= R(k) - \delta(k) Q(k); \\
S(k+1) &= \sum_{i=1}^{r} A_i^H R(k+1) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{R(k+1)} \overline{D_j}^H \\
&\quad + \sum_{s=1}^{k} \overline{H_s} R(k+1)^T \overline{G_s} + \sum_{l=1}^{u} N_l R(k+1)^H M_l \\
&= S(k) - \delta(k) \left(\sum_{i=1}^{r} A_i^H Q(k) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(k)} \overline{D_j}^H \right) \\
&\quad + \sum_{s=1}^{k} \overline{H_s} Q(k)^T \overline{G_s} + \sum_{l=1}^{u} N_l Q(k)^H M_l; \\
\gamma(k+1) &= \| S(k+1) \|^2; \\
\lambda(k) &= \gamma(k+1) / \gamma(k); \\
P(k+1) &= S(k+1) + \lambda(k) P(k);
\end{aligned}$$

Step 5. Go to step 3.

Some basic properties of algorithm 5.1 are listed in the following lemmas.

Lemma 5.4. For the sequences $S(k)$, $P(k)$ and $Q(k)$ which are generated by algorithm 5.1, if there exists a positive number r, such that $\|S(u)\| \neq 0$ and $\| Q(u)\| \neq 0, \forall u = 1, 2, \ldots, r$, then the following statements hold for $u, w = 1, 2, \ldots, r$ and $u \neq w$.

(1)$\langle S(u), S(w) \rangle = 0$,
(2)$\langle Q(u), Q(w) \rangle = 0$,
(3)$\langle P(u), S(w) \rangle = 0$.

Proof. Step 1: We prove the conclusion by induction. Because the real inner product is commutative, it is enough to prove three statements for $1 \leq u < w \leq r$.

For $u = 1, w = 2$, by applying of algorithm 5.1, we obtain

$$\begin{aligned}
&\langle S(1), S(2)\rangle \\
&= \Bigg\langle S(1), S(1) - \delta(1)\Bigg(\sum_{i=1}^{r} A_i^H Q(1) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(1)}\,\overline{D_j}^H\Bigg) \\
&\quad + \sum_{s=1}^{k} \overline{H_s} Q(1)^T \overline{G_s} + \sum_{l=1}^{u} N_l Q(1)^H M_l \Bigg\rangle \\
&= \|S(1)\|^2 - \delta(1)\Bigg\langle S(1), \sum_{i=1}^{r} A_i^H Q(1) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(1)}\,\overline{D_j}^H \\
&\quad + \sum_{s=1}^{k} \overline{H_s} Q(1)^T \overline{G_s} + \sum_{l=1}^{u} N_l Q(1)^H M_l \Bigg\rangle \\
&= \|S(1)\|^2 - \delta(1)\Bigg\langle \sum_{i=1}^{r} A_i S(1) B_i + \sum_{j=1}^{t} C_j \overline{S(1)} D_j \\
&\quad + \sum_{s=1}^{k} G_s S(1)^T H_s + \sum_{l=1}^{u} M_l S(1)^H N_l, Q(1) \Bigg\rangle \\
&= \|S(1)\|^2 - \delta(1)\langle Q(1), Q(1)\rangle = 0,
\end{aligned}$$

$$\begin{aligned}
&\langle Q(1), Q(2)\rangle \\
&= \Bigg\langle Q(1), \sum_{i=1}^{r} A_i P(2) B_i + \sum_{j=1}^{t} C_j \overline{P(2)} D_j). \\
&\quad + \sum_{s=1}^{k} G_s P(2)^T H_s + \sum_{l=1}^{u} M_l P(2)^H N_l \Bigg\rangle \\
&= \Bigg\langle Q(1), \sum_{i=1}^{r} A_i (S(2) + \lambda(1) P(1)) B_i + \sum_{j=1}^{t} C_j \overline{(S(2) + \lambda(1) P(1))} D_j \\
&\quad + \sum_{s=1}^{k} G_s (S(2) + \lambda(1) P(1))^T H_s + \sum_{l=1}^{u} M_l (S(2) + \lambda(1) P(1))^H N_l \Bigg\rangle \\
&= \lambda(1)\|Q(1)\|^2 + \Bigg\langle Q(1), \sum_{i=1}^{r} A_i S(2) B_i + \sum_{j=1}^{t} C_j \overline{S(2)} D_j \\
&\quad + \sum_{s=1}^{k} G_s S(2)^T H_s + \sum_{l=1}^{u} M_l S(2)^H N_l \Bigg\rangle \\
&= \lambda(1)\|Q(1)\|^2 - \frac{1}{\delta(1)}\Bigg\langle \sum_{i=1}^{r} A_i^H (R(2) - R(1)) B_i^H \\
&\quad + \sum_{j=1}^{t} \overline{C_j}^H \overline{(R(2) - R(1))}\,\overline{D_j}^H + \sum_{s=1}^{k} \overline{H_s} (R(2) \\
&\quad - R(1))^T \overline{G_s} + \sum_{l=1}^{u} N_l (R(2) - R(1))^H M_l, S(2) \Bigg\rangle \\
&= \lambda(1)\|Q(1)\|^2 - \frac{1}{\delta(1)}\langle S(2) - S(1), S(2)\rangle = 0,
\end{aligned}$$

and

$$\langle P(1), S(2)\rangle = \langle S(1), S(2)\rangle = 0.$$

Step 2: Now for $u<w<r$ we assume that

$$\langle S(u), S(w)\rangle = 0, \quad \langle Q(u), Q(w)\rangle = 0, \quad \langle P(u), S(w)\rangle = 0.$$

Thus, we can write

$$\begin{aligned}
&\langle S(u), S(w+1)\rangle \\
&= \Bigg\langle S(u), S(w) - \delta(w)\Bigg(\sum_{i=1}^{r} A_i^H Q(w) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H\Bigg) \\
&\quad + \sum_{s=1}^{k} \overline{H_s} Q(w)^T \overline{G_s} + \sum_{l=1}^{u} N_l Q(w)^H M_l \Bigg\rangle \\
&= -\delta(w)\Bigg\langle S(u), \sum_{i=1}^{r} A_i^H Q(w) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H \\
&\quad + \sum_{s=1}^{k} \overline{H_s} Q(w)^T \overline{G_s} + \sum_{l=1}^{u} N_l Q(w)^H M_l \Bigg\rangle \\
&= -\delta(w)\Bigg\langle \sum_{i=1}^{r} A_i S(u) B_i + \sum_{j=1}^{t} C_j \overline{S(u)} D_j + \sum_{s=1}^{k} G_s S(u)^T H_s \\
&\quad + \sum_{l=1}^{u} M_l S(u)^H N_l,\, Q(w) \Bigg\rangle \\
&= -\delta(w)\Bigg\langle \sum_{i=1}^{p} A_i (P(u) - \lambda(u-1)P(u-1)) B_i \\
&\quad + \sum_{j=1}^{t} C_j \overline{(P(u) - \lambda(u-1)P(u-1))} D_j \\
&\quad + \sum_{s=1}^{k} \overline{H_s} (P(u) - \lambda(u-1)P(u-1))^T \overline{G_s} \\
&\quad + \sum_{l=1}^{u} N_l (P(u) - \lambda(u-1)P(u-1))^H M_l,\, Q(w) \Bigg\rangle \\
&= -\delta(w)\langle Q(u) - \lambda(u-1)Q(u-1),\, Q(w)\rangle = 0,
\end{aligned}$$

$$\begin{aligned}
&\langle Q(u), Q(w+1)\rangle \\
&= \Bigg\langle Q(u), \sum_{i=1}^{r} A_i P(w+1) B_i + \sum_{j=1}^{t} C_j \overline{P(w+1)} D_j \\
&\quad + \sum_{s=1}^{k} G_s P(w+1)^T H_s + \sum_{l=1}^{u} M_l P(w+1)^H N_l \Bigg\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle Q(u), \sum_{i=1}^{r} A_i(S(w+1)+\lambda(w)P(w))B_i + \sum_{j=1}^{t} C_j\overline{(S(w+1)+\lambda(w)P(w))}D_j \right.\\
&\quad \left. + \sum_{s=1}^{k} G_s(S(w+1)+\lambda(w)P(w))^T H_s + \sum_{l=1}^{u} M_l(S(w+1)+\lambda(w)P(w))^H N_l \right\rangle\\
&= \left\langle Q(u), \sum_{i=1}^{r} A_i S(w+1)B_i + \sum_{j=1}^{t} C_j\overline{S(w+1)}D_j \right.\\
&\quad \left. + \sum_{s=1}^{k} G_s S(w+1)^T H_s + \sum_{l=1}^{u} M_l S(w+1)^H N_l + \lambda(w)Q(w) \right\rangle\\
&= -\frac{1}{\delta(u)}\left\langle R(u+1)-R(u), \sum_{i=1}^{r} A_i S(w+1)B_i + \sum_{j=1}^{t} C_j \right.\\
&\quad \left. \times\overline{S(w+1)}D_j + \sum_{s=1}^{k} G_s S(w+1)^T H_s + \sum_{l=1}^{u} M_l S(w+1)^H N_l \right\rangle\\
&= -\frac{1}{\delta(u)}\left\langle \sum_{i=1}^{r} A_i^H(R(u+1)-R(u))B_i^H + \sum_{j=1}^{t} \overline{C_j}^H\overline{(R(u+1)-R(u))}\,\overline{D_j}^H \right.\\
&\quad \left. + \sum_{s=1}^{k} \overline{H_s}(R(u+1)-R(u))^T\overline{G_s} + \sum_{l=1}^{u} N_l(R(u+1)-R(u))^H M_l, S(w+1) \right\rangle\\
&= -\frac{1}{\delta(u)}\langle S(u+1)-S(u), S(w+1)\rangle = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\langle P(u), S(w+1)\rangle\\
&\quad = \left\langle P(u), S(w)-\delta(w)\left(\sum_{i=1}^{r} A_i^H Q(w)B_i^H + \sum_{j=1}^{t} \overline{C_j}^H\overline{Q(w)}\,\overline{D_j}^H \right.\right.\\
&\qquad \left.\left. + \sum_{s=1}^{k} \overline{H_s}Q(w)^T\overline{G_s} + \sum_{l=1}^{u} N_l Q(w)^H\right) M_l \right\rangle\\
&\quad = -\delta(w)\left\langle P(u), \sum_{i=1}^{r} A_i^H Q(w)B_i^H + \sum_{j=1}^{t} \overline{C_j}^H\overline{Q(w)}\,\overline{D_j}^H \right.\\
&\qquad \left. + \sum_{s=1}^{k} \overline{H_s}Q(w)^T\overline{G_s} + \sum_{l=1}^{u} N_l Q(w)^H M_l \right\rangle\\
&\quad = -\delta(w)\left\langle \sum_{i=1}^{p} A_i P(u)B_i + \sum_{j=1}^{t} C_j\overline{P(u)}D_j \right.\\
&\qquad \left. + \sum_{s=1}^{k} G_s P(u)^T H_s + \sum_{l=1}^{u} M_l P(u)^H N_l, Q(w) \right\rangle = 0.
\end{aligned}
$$

For $u = w$, we have

$$\begin{aligned}
&\langle S(w), S(w+1)\rangle\\
&= \left\langle S(w), S(w) - \delta(w)\left(\sum_{i=1}^{r} A_i^H Q(w) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H\right.\right.\\
&\quad \left.\left. + \sum_{s=1}^{k} \overline{H_s} Q(w)^T \overline{G_s} + \sum_{l=1}^{u} N_l Q(w)^H M_l\right)\right\rangle\\
&= \|S(w)\|^2 - \delta(w)\left\langle S(w), \sum_{i=1}^{r} A_i^H Q(w) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H\right.\\
&\quad \left. \times \overline{Q(w)}\,\overline{D_j}^H + \sum_{s=1}^{k} \overline{H_s} Q(w)^T \overline{G_s} + \sum_{l=1}^{u} N_l Q(w)^H M_l\right\rangle\\
&= \|S(w)\|^2 - \delta(w)\left\langle \sum_{i=1}^{r} A_i S(w) B_i + \sum_{j=1}^{t} C_j \overline{S(w)} D_j\right.\\
&\quad \left. + \sum_{s=1}^{k} G_s S(w)^T H_s + \sum_{l=1}^{u} M_l S(w)^H N_l,\, Q(w)\right\rangle\\
&= \|S(w)\|^2 - \delta(w)\left\langle \sum_{i=1}^{r} A_i (P(w) - \lambda(w-1) P(w-1)) B_i\right.\\
&\quad + \sum_{j=1}^{t} C_j \overline{(P(w) - \lambda(w-1) P(w-1))} D_j\\
&\quad + \sum_{s=1}^{k} G_s (P(w) - \lambda(w-1) P(w-1))^T H_s\\
&\quad \left. + \sum_{l=1}^{u} M_l (P(w) - \lambda(w-1) P(w-1))^H N_l,\, Q(w)\right\rangle\\
&= \|S(w)\|^2 - \delta(w)\langle Q(w) - \lambda(w-1) Q(w-1),\, Q(w)\rangle = 0,
\end{aligned}$$

$$\begin{aligned}
&\langle Q(w),\, Q(w+1)\rangle\\
&= \left\langle Q(w), \sum_{i=1}^{r} A_i P(w+1) B_i + \sum_{j=1}^{t} C_j \overline{P(w+1)} D_j\right.\\
&\quad \left. + \sum_{s=1}^{k} G_s \|(w+1)^T H_s + \sum_{l=1}^{u} M_l P(w+1)^H N_l\right\rangle\\
&= \left\langle Q(w), \sum_{i=1}^{r} A_i (S(w+1) + \lambda(w) P(w)) B_i + \sum_{j=1}^{t} C_j \overline{(S(w+1) + \lambda(w) P(w))} D_j\right.\\
&\quad \left. + \sum_{s=1}^{k} G_s (S(w+1) + \lambda(w) P(w))^T H_s + \sum_{l=1}^{u} M_l (S(w+1) + \lambda(w) P(w))^H N_l\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle Q(w), \sum_{i=1}^{r} A_i S(w+1) B_i + \sum_{j=1}^{t} C_j \overline{S(w+1)} D_j) \right. \\
&\quad \left. + \sum_{s=1}^{k} G_s S(w+1)^T H_s + \sum_{l=1}^{u} M_l S(w+1)^H N_l + \lambda(w) Q(w) \right\rangle \\
&= \lambda(w) \| Q(w) \|^2 - \frac{1}{\delta(w)} \left\langle R(w+1) - R(w), \sum_{i=1}^{r} A_i S(w+1) B_i \right. \\
&\quad \left. + \sum_{j=1}^{t} C_j \overline{S(w+1)} D_j + \sum_{s=1}^{k} G_s S(w+1)^T H_s + \sum_{l=1}^{u} M_l S(w+1)^H N_l \right\rangle \\
&= \lambda(w) \| Q(w) \|^2 - \frac{1}{\delta(w)} \left\langle \sum_{i=1}^{r} A_i^H (R(w+1) - R(w)) B_i^H \right. \\
&\quad + \sum_{j=1}^{t} \overline{C_j}^H \overline{(R(w+1) - R(w))} \overline{D_j}^H + \sum_{s=1}^{k} \overline{H_s} (R(w+1) - R(w))^T \overline{G_s} \\
&\quad \left. + \sum_{l=1}^{u} N_l (R(w+1) - R(w))^H M_l, S(w+1) \right\rangle \\
&= \lambda(w) \| Q(w) \|^2 - \frac{1}{\delta(w)} \langle S(w+1) - S(w), S(w+1) \rangle \\
&= \lambda(w) \| Q(w) \|^2 - \frac{1}{\delta(w)} \| S(w+1) \|^2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\langle P(w), S(w+1) \rangle \\
&\quad = \left\langle P(w), S(w) - \delta(w) \left(\sum_{i=1}^{r} A_i^H Q(w) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)} \overline{D_j}^H \right. \right. \\
&\qquad \left. \left. + \sum_{s=1}^{k} \overline{H_s} Q(w)^T \overline{G_s} + \sum_{l=1}^{u} N_l Q(w)^H M_l \right) \right\rangle \\
&\quad = \langle S(w) + \lambda(w-1) S(w-1) + \lambda(w-1)\lambda(w-2) S(w-2) + \cdots \\
&\qquad + \lambda(w-1) \ldots \lambda(1) S(1), S(w) \rangle - \delta(w) \left\langle P(w), \sum_{i=1}^{r} A_i^H Q(w) B_i^H \right. \\
&\qquad \left. + \sum_{j=1}^{t} \overline{C_j}^H \overline{Q(w)} \overline{D_j}^H + \sum_{s=1}^{k} \overline{H_s} Q(w)^T \overline{G_s} + \sum_{l=1}^{u} N_l Q(w)^H M_l \right\rangle \\
&\quad = \| S(w) \|^2 - \delta(w) \left\langle \sum_{i=1}^{r} A_i P(w) B_i + \sum_{j=1}^{t} C_j \overline{P(w)} D_j \right. \\
&\qquad \left. + \sum_{s=1}^{k} G_s P(w)^T H_s + \sum_{l=1}^{u} M_l P(w)^H N_l, Q(w) \right\rangle = 0.
\end{aligned}
$$

By the principle of induction, we draw the conclusion. □

Lemma 5.5. If there exists a positive number l such that $\delta(l) = 0$ or $\delta(l) = \infty$ in algorithm 5.1, then $X(l)$ is a solution of problem 5.1.

Proof. If $\delta(l) = 0$, we have $\|S(l)\|^2 = 0$. If $\delta(l) = \infty$, we have $\|Q(l)\|^2 = 0$. It is showed that

$$\begin{aligned}
&\|S(l)\|^2 \\
&\quad = \langle S(l) + \lambda(l-1)S(l-1) + \lambda(l-1)\lambda(l-2)S(l-2) + \cdots \\
&\qquad + \lambda(l-1)\ldots\lambda(1)S(1), S(l)\rangle \\
&\quad = \langle P(l), S(l)\rangle = \left\langle P(l), \sum_{i=1}^{p} A_i^H R(l) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{R(l)}\,\overline{D_j}^H \right\rangle \\
&\quad = \left\langle \sum_{i=1}^{p} A_i P(l) B_i + \sum_{j=1}^{t} C_j \overline{P(l)} D_j, R(l) \right\rangle = 0.
\end{aligned}$$

Therefore, for $\delta(l) = 0$ or $\delta(l) = \infty$ we have

$$S(l) = \sum_{i=1}^{p} A_i^H R(l) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{R(l)}\,\overline{D_j}^H = 0.$$

So by lemma 5.2 we can conclude that $X(l)$ is the solution of problem 5.1. ☐

Theorem 5.1. If the matrix equation (5.0.6) is consistent, then, for any arbitrary initial matrix $X(l)$, the solution X^* of the problem 5.1 can be obtained by using algorithm 5.1 within a finite iterative steps in the absence of roundoff errors.

Proof. By lemma 5.4, the set $S(i), i = 1, 2, 3, \ldots, m \times n$ is an orthogonal basis of the real inner product space $\mathfrak{C}^{m\times n}$ with dimension $m \times n$. Therefore, we can obtain $\|S(m \times n + 1)\| = 0$. It is also showed that $X(m \times n + 1)$ is the solution of problem 5.1 in the absence of roundoff errors. ☐

Theorem 5.2. Supposed that the initial matrix is $\sum_{i=1}^{p} A_i^H F(l) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{F(l)}\,\overline{D_j}^H$, where $F(1) \in \mathfrak{C}^{m\times n}$ is an arbitrary matrix, or especially $X(1) = 0$, then the solution X^* generated by algorithm 5.1 is the least norm solution of problem 5.1.

Proof. Let the initial matrix be $X(1) = \sum_{i=1}^{p} A_i^H F(l) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{F(l)}\ \overline{D_j}^H$, then, it follows from algorithm 5.1 that the generated matrix $X(k)$ can be easily expressed as $X(k) = \sum_{i=1}^{p} A_i^H F(k) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{F(k)}\,\overline{D_j}^H$ for certain matrices $F(k) \in \mathfrak{C}^{m\times n}$ for $k = 2, 3, \ldots$. This implies that there exists a matrix $F^* \in \mathfrak{C}^{m\times n}$ such that $X^* = \sum_{i=1}^{p} A_i^H F^* B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{F^*}\,\overline{D_j}^H$. Now if we let $\tilde{X}^*$ is an arbitrary solution of problem 5.1, then, by using lemma 5.3, there exists the matrix $Z^* \in \mathfrak{C}^{m\times n}$ such that $\tilde{X}^* = X^* + Z^*$ and $\sum_{i=1}^{p} A_i Z^* B_i + \sum_{j=1}^{t} C_j \overline{Z^*} D_j = 0$. Thus, one can obtain

$$
\begin{aligned}
&\langle X^*, Z^* \rangle \\
&\quad = \left\langle \sum_{i=1}^{p} A_i^H F^* B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{F^*} \overline{D_j}^H, Z^* \right\rangle \\
&\quad = \left\langle F^*, \sum_{i=1}^{p} A_i Z^* B_i + \sum_{j=1}^{t} C_j \overline{Z^*} D_j \right\rangle = 0.
\end{aligned}
$$

By applying the relation above, one can obtain

$$
\begin{aligned}
\|\tilde{X}^*\|^2 &= \|X^* + Z^*\|^2 \\
&= \|X^*\|^2 + \|Z^*\|^2 + 2\langle X^*, Z^* \rangle = \|X^*\|^2 + \|Z^*\|^2 \geq \|X^*\|^2.
\end{aligned}
$$

This shows that the solution X^* is the least Frobenius norm solution of problem 5.1. □

Similar to[98], the minimization property of the proposed algorithm is stated as follows. This property shows that algorithm 5.1 converges smoothly.

Theorem 5.3. For any initial matrix $X(1) \in \mathfrak{C}^{m\times n}$, we have

$$
\begin{aligned}
&\Bigg\| \sum_{i=1}^{r} A_i X(k+1) B_i + \sum_{j=1}^{t} C_j \overline{X(k+1)} D_j \\
&\qquad + \sum_{s=1}^{k} G_s X(k+1)^T H_s + \sum_{l=1}^{u} M_l X(k+1)^H N_l - E \Bigg\|^2 \\
&\qquad = \min_{X \in \psi_k} \left\| \sum_{i=1}^{r} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j + \sum_{s=1}^{k} G_s X^T H_s + \sum_{l=1}^{u} M_l X^H N_l - E \right\|^2,
\end{aligned}
\tag{5.1.1}
$$

where $X(k+1)$ is generated by algorithm 5.1 at the $k+1$-th iteration and ψ_k presents an affine subspace which has the following form

$$
\psi_k = X(1) + \mathrm{span}(P(1), P(2), \ldots, P(k)). \tag{5.1.2}
$$

Proof. For any matrix $X \in \psi_k$, it follows from equation (5.1.2) that there exist numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$
X = X(1) + \sum_{l=1}^{k} \alpha_l P(l).
$$

Now we define the continuous and differentiable function f with respect to the variable $\alpha_1, \alpha_2, \ldots, \alpha_k$ as

$$
\begin{aligned}
&f(\alpha_1, \alpha_2, \ldots, \alpha_k)\\
&\quad= \Bigg\| \sum_{i=1}^{r} A_i(X(1) + \sum_{l=1}^{k} \alpha_l P(l))B_i + \sum_{j=1}^{t} C_j\overline{(X(1) + \sum_{l=1}^{k} \alpha_l P(l))}D_j\\
&\qquad + \sum_{s=1}^{k} G_s(X(1) + \sum_{l=1}^{k} \alpha_l P(l))^T H_s + \sum_{l=1}^{u} M_l(X(1) + \sum_{l=1}^{k} \alpha_l P(l))^H N_l - E \Bigg\|^2.
\end{aligned}
$$

Due to lemma 5.4 one can obtain

$$
\begin{aligned}
&f(\alpha_1, \alpha_2, \ldots, \alpha_k)\\
&\quad= \Bigg\| \sum_{i=1}^{r} A_i X(1) B_i + \sum_{j=1}^{t} C_j \overline{X(1)} D_j + \sum_{s=1}^{k} G_s X(1)^T H_s + \sum_{l=1}^{u} M_l X(1)^H N_l - E\\
&\qquad + \sum_{l=1}^{k} \alpha_l [\sum_{i=1}^{r} A_i P(l) B_i + \sum_{j=1}^{t} C_j \overline{P(l)} D_j + \sum_{s=1}^{k} G_s P(l)^T H_s + \sum_{l=1}^{u} M_l P(l)^H N_l] \Bigg\|^2\\
&\quad= \|R(1)\|^2 + \sum_{l=1}^{k} \alpha_l^2 \|Q(1)\|^2 - 2\alpha_l \langle Q(1), R(1)\rangle.
\end{aligned}
$$

Now we consider the problem of minimizing the function $f(\alpha_1, \alpha_2, \ldots, \alpha_k)$. It is obvious that

$$
\begin{aligned}
&\min_{\alpha_l} f(\alpha_1, \alpha_2, \ldots, \alpha_k)\\
&\quad= \min_{X \in \psi_k} \Bigg\| \sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j + \sum_{s=1}^{k} G_s X^T H_s + \sum_{l=1}^{u} M_l X^H N_l - E \Bigg\|^2.
\end{aligned}
$$

According to this function, the minimum occurs when

$$
\frac{\partial(\alpha_1, \alpha_2, \ldots, \alpha_k)}{\partial \alpha_l} = 0 \quad \text{for} \quad l = 1, 2, \ldots, k.
$$

Consequently, we can get

$$
\alpha_l = \frac{\langle Q(l), R(l)\rangle}{\|Q(l)\|^2}.
$$

Since algorithm 5.1, one can obtain

$$
R(1) = R(l) + \delta(l-1)Q(l-1) + \delta(l-2)Q(l-2) + \cdots + \delta(1)Q(1).
$$

Therefore, it follows from lemma 5.4 that

$$\begin{aligned}\alpha_l &= \frac{\langle Q(l), R(l)\rangle}{\| Q(l)\|^2}\\ &= \frac{\left\langle P(l), \sum_{i=1}^{r} A_i^H R(l) B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{R(l)}\,\overline{D_j}^H + \sum_{s=1}^{k} \overline{H_s} R(l)^T \overline{G_s} + \sum_{l=1}^{u} N_l R(l)^H M_l \right\rangle}{\| Q(l)\|^2}\\ &= \frac{\langle P(l), S(l)\rangle}{\| Q(l)\|^2} = \frac{\langle S(l) + \lambda(l-1)P(l-1), S(l)\rangle}{\| Q(l)\|^2} = \frac{\| S(l)\|^2}{\| Q(l)\|^2} = \delta(l).\end{aligned}$$

By this means, we complete the proof. □

By theorem 5.1, the solution generalized by algorithm 5.1 at the $k+1$-th iteration for any initial matrix minimizes the residual norm in the affine subspace ψ_k. Also one has

$$\begin{aligned}&\left\| \sum_{i=1}^{s} A_i X(k+l) B_i + \sum_{j=1}^{t} C_j \overline{X(k+l)} D_j + \sum_{s=1}^{k} G_s X(k+l)^T H_s \right.\\ &\quad \left. + \sum_{l=1}^{u} M_l X(k+l)^H N_l - E \right\|^2\\ &\qquad \leq \left\| \sum_{i=1}^{s} A_i X(k) B_i + \sum_{j=1}^{t} C_j \overline{X(k)} D_j + \sum_{s=1}^{k} G_s X(k)^T H_s + \sum_{l=1}^{u} M_l X(k)^H N_l - E \right\|^2,\end{aligned} \tag{5.1.3}$$

which shows that the sequence of the norm of residuals $\|R(1)\|, \|R(2)\|, \ldots$ is monotonically decreasing. This decent property of the norm of the residuals shows that the algorithm 5.1 processes fast and smoothly convergence.

Now we will solve problem 5.2. For a given matrix X_0 by problem 5.1, we can get

$$\begin{aligned}&\min_{X\in C^{m\times n}} \left\| \sum_{i=1}^{r} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j + \sum_{s=1}^{k} G_s X^T H_s + \sum_{l=1}^{u} M_l X^H N_l - E \right\|\\ &\quad = \min_{X\in C^{m\times n}} \left\| \sum_{i=1}^{r} A_i (X - X_0) B_i + \sum_{j=1}^{t} C_j \overline{(X - X_0)} D_j \right.\\ &\qquad + \sum_{s=1}^{k} G_s (X - X_0)^T H_s + \sum_{l=1}^{u} M_l (X - X_0)^H N_l\\ &\qquad \left. - \left(E - \sum_{i=1}^{r} A_i X_0 B_i - \sum_{j=1}^{t} C_j \overline{X_0} D_j + \sum_{s=1}^{k} G_s X_0^T H_s + \sum_{l=1}^{u} M_l X_0^H N_l \right) \right\|.\end{aligned}$$

If we let the set $E_1 = E - \sum_{i=1}^{p} A_i X_0 B_i - \sum_{j=1}^{t} C_j \overline{X_0} D_j - \sum_{s=1}^{k} G_s X_0^T H_s - \sum_{l=1}^{u} M_l X_0^H N_l$ and $X_1 = X - X_0$, then problem 5.2 is equivalent to find the least Frobenius norm solution X_1^* of

$$\min_{X_1 \in C^{m\times n}} \left\| \sum_{i=1}^{r} A_i X_1 B_i + \sum_{j=1}^{t} C_j \overline{X_1} D_j + \sum_{s=1}^{k} G_s X_1^T H_s + \sum_{l=1}^{u} M_l X_1^H N_l - E_1 \right\|,$$

which can be computed by using algorithm 5.1 with the initial matrix $X_1(1) = \sum_{i=1}^{s} A_i^H F B_i^H + \sum_{j=1}^{t} \overline{C_j}^H \overline{F} \overline{D_j^H} + \sum_{s=1}^{k} \overline{H_s} F^T \overline{G_s} + \sum_{l=1}^{u} N_l F^H M_l$ where $F \in \mathfrak{C}^{m\times n}$ is an arbitrary matrix, or especially $X_1(1) = 0$. Thus the solution of problem 5.2 $\|\check{X} - X_0\| = \min_{X \in G_r} \|X - X_0\|$ can be stated as

$$\check{X} = X_1^* + X_0.$$

It's showed that we can solve the problem 5.2 based on the problem 5.1. In other words, we can obtain the solution of problem 5.2 by using the solution of problem 5.1.

5.2 Numerical Example

In this section, in order to illustrate the efficiency of algorithm 5.1, two numerical examples are given. All the tests are performed by MATLAB with machine precision around 10^{-16}.

Example 5.1. Find the least Frobenius norm solution of the following matrix equation

$$AXB + C\overline{X}D + EX^H F + GX^T H = M, \tag{5.2.1}$$

where

$$M = \begin{pmatrix} 5176 + 1755i & 10683 + 3023i & 3277 + 5921i & 10579 + 4432i \\ 30638 + 224i & 20539 - 885i & 21270 - 454i & 23668 + 8249i \\ 49038 - 1619i & 33221 - 13413i & 39726 - 10567i & 34533 + 1371i \\ 15249 + 15652i & 21330 + 7577i & 5458 + 8704i & 14865 + 13306i \end{pmatrix},$$

$$A = \begin{pmatrix} 1+i & 1-i & 6i & 4+3i \\ 2+i & 0 & 12 & 10 \\ 5+6i & 2+3i & 11-2i & i \\ 1 & 12 & 0 & 9i \end{pmatrix}, C = \begin{pmatrix} i & 9 & 15 & 2i \\ 11 & 2+3i & 12 & 11 \\ 18 & 11-9i & 21 & i \\ 9+8i & 11 & i & 8i \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 3-2i & 4+6i & 9+8i \\ 4+6i & 11 & 12 & 9 \\ 0 & 12 & 15 & 18 \\ 2-i & -9i & 12 & 11 \end{pmatrix}, E = \begin{pmatrix} -i & 8 & 1 & 2i \\ 11 & 21-3i & 12 & 11 \\ 78 & 11 & 21 & i \\ 9-8i & 11 & i & -8i \end{pmatrix},$$

$$H=\begin{pmatrix} 3 & 9-i & 15 & 2+i \\ 11 & 2+3i & 1+2i & 11 \\ 18 & 11+9i & 21 & i \\ 8i & 21 & -i & 8-i \end{pmatrix}, F=\begin{pmatrix} i & 9 & 1-5i & 2i \\ 11 & 2+3i & 12 & 11 \\ 18 & -9i & 21 & i \\ 9+8i & 1-8i & 9 & 0 \end{pmatrix},$$

$$D=G=\begin{pmatrix} 9+2i & -i & i & 6+8i \\ 4i & 19 & 0 & 11 \\ 23 & 26+i & 9i & 9 \\ 23 & 0 & 6 & 8i \end{pmatrix}.$$

It can be verified that the generalized Sylvester conjugate transpose matrix equation (5.2.1) are consistent and have the solution

$$X=\begin{pmatrix} 4+3i & 2+i & 11 & 6 \\ 9+6i & 11+2i & 8+9i & 9 \\ 9i & 4 & 2i & 11i \\ 23 & 4 & 7 & 12 \end{pmatrix}.$$

If we let the initial matrix $X_1=0$, by using algorithm 5.1, we obtain the solution

$$X(133)=\begin{pmatrix} 4.0000+3.0000i & 2.0000+1.0000i & 10.9999 & 5.9999 \\ 8.9999+6.0000i & 11.0000+2.0000i & 8.0000+9.0000i & 9.0000 \\ 8.9999i & 4.0000 & 1.9999i & 11.0000i \\ 22.9999 & 3.9999 & 7.0000 & 12.0000 \end{pmatrix},$$

with corresponding residual

$$r(133)=4.7116\times 10^{-16},$$

and relative error

$$e(133)=2.6266\times 10^{-15}.$$

The obtained results are presented in figures 5.2.1 and 5.2.2,where

$$r_k=\log_{10}\|M-AX(k)B-C\overline{X(k)}D-EX(k)^HF-GX(k)^TH\|,$$

and

$$\operatorname{err}(k)=\frac{\|X(k)-X\|}{\|X\|}.$$

If we let the initial matrix $X(1)=0, X(1)=I_4, X(1)=\text{ones}(4,4)$, respectively, according to algorithm 5.1, the residual and the relative error of solution are presented in figures 5.2.3 and 5.2.4.

Remark 1. From figures 5.2.1 and 5.2.2, the residual and the relative error of the solution to matrix equation (5.2.1) are reduced with the addition of an iterative number. It showed that the iterative solution converges to the exact solution.

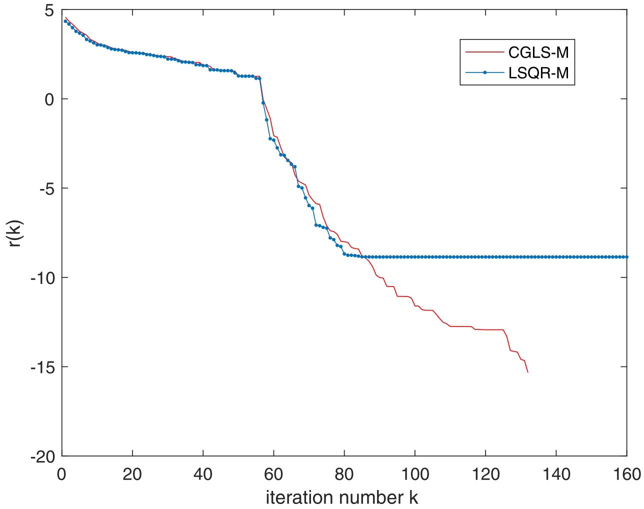

FIG. 5.2.1 – The residual of solution for example 5.1.

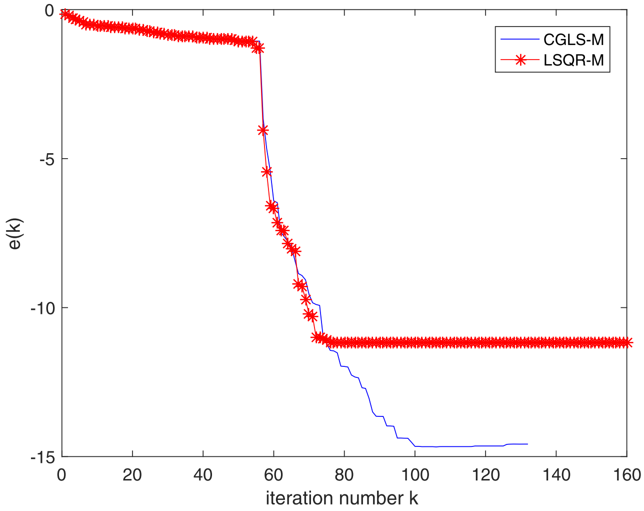

FIG. 5.2.2 – The relative error of solution for example 5.1.

Moreover, with the addition of an iterative number, the residual and the relative error of the LSQR-M iterative algorithm are not varied. Finally, the residual and the relative error of the iterative algorithm are smaller than those of the LSQR-M iterative algorithm. It is proved that our proposed method (MCGLS iterative algorithm or CGLS-M iterative algorithm) is efficient and is a good algorithm.

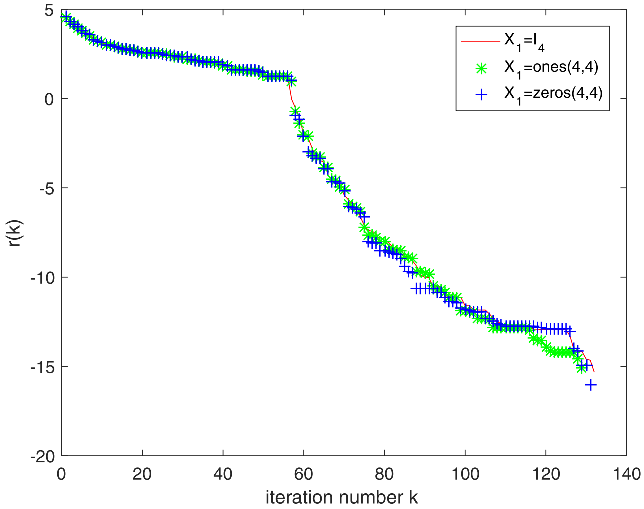

FIG. 5.2.3 – The relative error of solution for different initial values for example 5.1.

Remark 2. From figure 5.2.3, for the different initial value $X(1) = 0$, $X(1) = I_4, X(1) = \text{ones}(4,4)$, respectively, the iterative solution to the matrix equation (5.2.1) all converges to the exact solution to the matrix equation (5.2.1).

Example 5.2. Find the least Frobenius norm solution of the following matrix equation

$$AXB + C\overline{X}D + EX^HF + GX^TH = M, \tag{5.2.2}$$

where

$$A = \text{rand}(6,7) + i * \text{rand}(6,7), B = \text{rand}(5,4) + i * \text{rand}(5,4),$$

$$C = \text{rand}(6,7) + i * \text{rand}(6,7), D = \text{rand}(5,4) + i * \text{rand}(5,4),$$

$$E = \text{rand}(6,5) + i * \text{rand}(6,5), F = \text{rand}(7,4) + i * \text{rand}(7,4),$$

$$G = \text{rand}(6,5) + i * \text{rand}(6,5), H = \text{rand}(7,4) + i * \text{rand}(7,4),$$

$$M = \begin{pmatrix} -1.1212 + 36.8478i & 0.2336 + 32.7105i & 0.4078 + 32.5045i & 5.5394 + 29.9953i \\ -5.7270 + 32.8779i & -8.4829 + 28.7701i & -4.6524 + 30.6806i & -1.1100 + 27.3557i \\ -1.1063 + 37.1070i & -2.1256 + 35.1041i & 1.4087 + 34.2908i & 5.4257 + 30.3026i \\ -1.2607 + 24.4888i & -4.0543 + 21.8070i & 0.8941 + 24.5297i & 3.5789 + 20.1358i \\ -2.9581 + 34.7771i & -4.3860 + 31.0165i & -1.5855 + 33.7016i & 1.2014 + 28.1810i \\ -6.5203 + 25.7637i & -7.0275 + 22.1274i & -5.4346 + 24.7957i & 0.5099 + 20.4696i \end{pmatrix}.$$

It can be verified that the generalized Sylvester conjugate transpose matrix equation (5.2.2) are consistent and have the solution

$$X = \left(\begin{matrix} 0.6273 + 0.6164i & 0.6173 + 0.9883i & 0.8044 + 0.5278i \\ 0.0216 + 0.9397i & 0.5754 + 0.7668i & 0.9861 + 0.4116i \\ 0.9106 + 0.3545i & 0.5301 + 0.3367i & 0.0299 + 0.6026i \\ 0.8006 + 0.4106i & 0.2751 + 0.6624i & 0.5357 + 0.7505i \\ 0.7458 + 0.9843i & 0.2486 + 0.2442i & 0.0870 + 0.5835i \\ 0.8131 + 0.9456i & 0.4516 + 0.2955i & 0.8021 + 0.5518i \\ 0.3833 + 0.6766i & 0.2277 + 0.6802i & 0.9891 + 0.5836i \end{matrix} \right.$$

$$\left. \begin{matrix} 0.0669 + 0.5118i & 0.8990 + 0.8865i \\ 0.9394 + 0.0826i & 0.6259 + 0.4547i \\ 0.0182 + 0.7196i & 0.1379 + 0.4134i \\ 0.6838 + 0.9962i & 0.2178 + 0.2177i \\ 0.7837 + 0.3545i & 0.1821 + 0.1257i \\ 0.5341 + 0.9713 & 0.0418 + 0.3089 \\ 0.8854 + 0.3464i & 0.1069 + 0.7261i \end{matrix} \right).$$

If we let the initial matrix $X(1) = 0$, according to algorithm 5.1, we obtain the solutions with corresponding residual

$$r(111) = 5.4178 \times 10^{-16}.$$

The obtained results are presented in figure 5.2.4, where

$$r_k = \log_{10} \| M - AX(k)B - C\overline{X(k)}D - EX(k)^H F - GX(k)^T H \|.$$

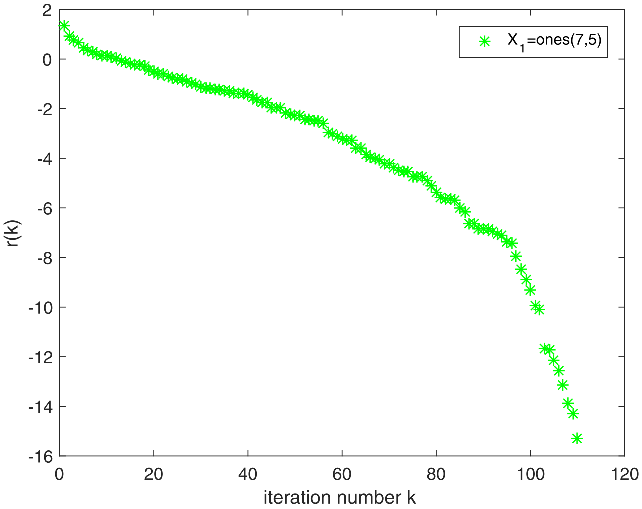

FIG. 5.2.4 – The residual of solution for example 5.2.

Remark 3. Figure 5.2.4 showed that the iterative solution converges to the exact solution for any random coefficient matrices. Moreover, with the addition of an iterative number, the residual of the iterative algorithm is smaller. It is proved that our proposed method (CGLS-M iterative algorithm) is feasible and is a magnificent algorithm.

5.3 Conclusions

In order to solve the least Frobenius norm solution of the generalized Sylvester conjugate transpose matrix equation (5.0.6), we constructed the CGLS-M iterative algorithm in this chapter by extending the classical CGLS iterative algorithm. Due to this newly presented iterative algorithm, for any initial matrix $X(1)$, a solution X^* can be obtained in finite iteration steps in the absence of roundoff errors, what is more, the least Frobenius norm solution can also be obtained by choosing any initial matrix. Besides, by use of this iterative method, if a matrix X_0 is given, then the optimal approximation solution $\check{X}$ can be derived by first finding the least Frobenius norm solution of a new corresponding matrix equation. Furthermore, two numerical examples are given to check the obtained theory. It is worth noting that the coefficient matrices of the second numerical example are random matrices. It shows that the proposed iterative algorithm is quite more efficient than the existing LSQR iterative algorithm[90,93,94].

Chapter 6

MCGLS Iterative Algorithm to Coupled Linear Operator Systems

In this chapter, we investigate the least squares problem of the following coupled linear operator matrix equations

$$\begin{cases} L_{11}(X) + L_{12}(Y) = E_1, \\ L_{21}(X) + L_{22}(Y) = E_2, \end{cases} \tag{6.0.1}$$

in which $L_{ij}, i, j = 1, 2$. Denotes a real linear operator and $X, Y \in \mathfrak{C}^{m\times n}, E_i \in \mathfrak{C}^{p\times q}, i = 1, 2$. Obviously, the matrix equations (6.0.1) includes the general matrix equations

$$\begin{aligned} &\sum_{m=1}^{r} A_{im} X B_{im} + \sum_{j=1}^{t} C_{ij} \overline{X} D_{ij} + \sum_{s=1}^{k} G_{is} X^H H_{is} \\ &+ \sum_{h=1}^{g} G_{ih} Y^T H_{ih} + \sum_{l=1^u} M_{il} Y N_{il} = E_i, i = 1, 2. \end{aligned}$$

as special case. So this chapter aims to propose the least squares iterative solution of coupled linear operator matrix equations and generalize the results of the paper in[90]. In this chapter, we mainly consider the following two problems.

Problem 6.1. For given $E_i \in \mathfrak{C}^{p\times q}, i = 1, 2$; find $X^*, Y^* \in \mathfrak{C}^{m\times n}$ such that

$$\left\| \begin{pmatrix} L_{11}(X^*) + L_{12}(Y^*) - E_1 \\ L_{21}(X^*) + L_{22}(Y^*) - E_2 \end{pmatrix} \right\| = \min. \tag{6.0.2}$$

Problem 6.2. Let G_r denote the solution set of problem 6.1, for given $X_0, Y_0 \in \mathfrak{C}^{m\times n}$, find the matrix $\check{X} \in S_r$, such that

$$\|\check{X} - X_0\| = \min_{\mathrm{X}\in G_r} \|X - X_0\|, \quad \|\check{Y} - Y_0\| = \min_{Y\in G_r} \|Y - Y_0\|. \tag{6.0.3}$$

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6.1 Some Useful Lemmas

Two matrices X and Y are said to be orthogonal if $\langle X, Y\rangle = 0$. For a linear operator L and matrices B, X with appropriate dimension, a well-known property of the inner product is $\langle L(X), B\rangle = \langle X, L^*(B)\rangle$, in which the symbol L^* is the conjugate operator of L.

Lemma 6.1. [99]Let U be an inner product space, V be a subspace of U, and $V^\perp$ be the orthogonal complement subspace of V. Given $w \in U$, if there exists a $w_0 \in V$ such that $\|w - v_0\| \le \|w - v\|$ holds for any $v \in V$, then v_0 is unique and $v_0 \in V$ is the unique minimization vector in V if and only if $(w - v_0) \perp V$, *i.e.*$(w - v_0) \in V^\perp$.

Lemma 6.2. If we let $\hat{R}_i = E_i - L_{i1}(\hat{X}) - L_{i2}(\hat{Y}), i = 1, 2$ be the residual of equations (6.0.1) corresponding to the matrices $\hat{X}, \hat{Y} \in C^{m\times n}$, then the matrices $\hat{X}$ and $\hat{Y}$ are the least squares solutions of equation (6.0.1) if

$$L_{i1}^*(\hat{R}_i) = 0, \quad L_{i2}^*(\hat{R}_i) = 0, \quad i = 1, 2. \tag{6.1.1}$$

Proof. Firstly, the linear subspace can be defined as $W_i = \{F_i | F_i = L_{i1}(X) + L_{i2}(Y)\}$, $i = 1, 2$ with $F_i \in \mathfrak{C}^{p\times q}$ and $X \in \mathfrak{C}^{m\times n}$. Next, if we suppose that $\hat{F}_i = L_{i1}(\hat{X}) + L_{i2}(\hat{Y}), i = 1, 2,$ then we have $\hat{F}_i \in W_i, i = 1, 2$. Thus, for any matrices $F_i \in W_i, i = 1, 2$, we have

$$\begin{aligned}\langle E_i - \hat{F}_i, F_i\rangle &= \langle E_i - L_{i1}(\hat{X}) - L_{i2}(\hat{Y}), L_{i1}(X) + L_{i2}(Y)\rangle \\ &= \langle \hat{R}_i, L_{i1}(X) + L_{i2}(Y)\rangle \\ &= \langle \hat{R}_i, L_{i1}(X)\rangle + \langle \hat{R}_i, L_{i2}(Y)\rangle \\ &= \langle L_{i1}(\hat{R}_i), X\rangle + \langle L_{i2}(\hat{R}_i), Y\rangle.\end{aligned}$$

Therefore, let $(L_{i1}^*(\hat{R}_i), L_{i2}^*(\hat{R}_i)) = 0, i = 1, 2,$ then we have $\langle E_i - \hat{F}_i, F_i\rangle = 0, i = 1, 2$. According to lemma 6.1, we can get $(E_i - \hat{F}_i) \in W_i^\perp, i = 1, 2$. So matrix $\hat{X}$ is the least squares solution of equations (6.0.1). □

Lemma 6.3. Supposed that $(\hat{X}, \hat{Y})$ be the solutions of problem 6.1, then any solutions $(\tilde{X}, \tilde{Y})$ of problem 6.1 can be characterized by $(\hat{X} + Z, \hat{Y} + Z)$ in which matrix $Z \in \mathfrak{C}^{m\times n}$ satisfies

$$L_{i1}(Z) + L_{i2}(Z) = 0, \quad i = 1, 2. \tag{6.1.2}$$

Proof. For any solution $(\tilde{X}, \tilde{Y})$ of problem 6.1, if the matrix defined as $Z = \tilde{X} - \hat{X}$, then we have $\tilde{X} = \hat{X} + Z$. From lemma 6.2, we have

$$\begin{aligned}
&\|L_{i1}(\hat{X})+L_{i2}(\hat{Y})-E_i\|^2\\
&\quad=\|L_{i1}(\tilde{X})+L_{i1}(\tilde{Y})-E_i\|^2\\
&\quad=\|L_{i1}(\hat{X}+Z)+L_{i2}(\hat{Y}+Z)-E_i\|^2\\
&\quad=\|L_{i1}(Z)+L_{i2}(Z)-\hat{R}_i\|^2\\
&\quad=\|L_{i1}(Z)+L_{i2}(Z)\|^2+\|\hat{R}_i\|^2-2\langle L_{i1}(Z)+L_{i2}(Z),\hat{R}_i\rangle\\
&\quad=\|L_{i1}(Z)+L_{i2}(Z)\|^2+\|\hat{R}_i\|^2-2\langle Z,L_{i1}^*(\hat{R}_i)+L_{i2}^*(\hat{R}_i)\rangle\\
&\quad=\|\hat{R}_i\|^2.
\end{aligned}$$

Thus, we complete the proof of the conclusion. □

6.2 MCGLS Iterative Algorithm and Convergence Analysis

In this section, we first present a CGLS-M iterative algorithm, then we give some basic properties of the presented CGLS-M or MCGLS iterative algorithm. At the end of this section, we put the answer to find the least Frobenius norm solution of problem 6.1.

Algorithm 6.1

Step 1. Input complex matrices $E_i \in C^{p\times q}, i=1,2$ and initial matrices $X(1)$, $Y(1)\in\mathfrak{C}^{m\times n}$;
Step 2. Compute

$$\begin{aligned}
R_i(1)&=E_i-L_{i1}(X(1))-L_{i2}(Y(1)), i=1,2,\\
T_i(1)&=L_{i1}^*(R_i(1))+L_{i2}^*(R_i(1)), i=1,2,\\
P_i(1)&=T_i(1), i=1,2,\\
\gamma(1)&=\|T_1(1)\|^2+\|T_2(1)\|^2;
\end{aligned}$$

Step 3. For $k=1,2,3,\ldots$ repeat the following, if $\|R_1(k)\|+\|R_2(k)\|=0$, then stop and $X(k)$, $Y(k)$ are the solutions of equations (6.0.1), break;
else if $\|R_1(k)\|+\|R_2(k)\|\neq 0$ but $\|T_1(k)\|+\|T_2(k)\|=0$, then stop and $X(k)$, $Y(k)$ are the solutions of problem 6.1, break;
else $k:=k+1$;
Step 4. Compute

$$\begin{aligned}
Q_i(k)&=L_{i1}(P_i(k))+L_{i2}(P_i(k)), i=1,2,\\
\delta(k)&=\gamma(k)/(\|Q_1(k)\|^2+\|Q_2(k)\|^2),
\end{aligned}$$

$$\begin{aligned}
X(k+1) &= X(k) + \frac{1}{2}\delta(k)(P_1(k) + P_2(k)),\\
Y(k+1) &= Y(k) + \frac{1}{2}\delta(k)(P_1(k) + P_2(k)),\\
R_i(k+1) &= R_i(k) - \delta(k)Q_i(k), i = 1, 2,\\
T_i(k+1) &= L_{i1}^*(R_i(k+1)) + L_{i2}^*(R_i(k+1))\\
&= T_i(k) - \delta(k)(L_{i1}^*(Q_i(k) + L_{i2}^*(Q_i(k))), i = 1, 2,\\
\gamma(k+1) &= \|T_1(k+1)\|^2 + \|T_2(k+1)\|^2,\\
\lambda(k) &= \gamma(k+1)/\gamma(k),\\
P_i(k+1) &= T_i(k+1) + \lambda(k)P_i(k), i = 1, 2;
\end{aligned}$$

Step 5. Go to step 3.

In the following, we give some basic properties of algorithm 6.1.

Lemma 6.4. If there exists a positive number r, such that $\|T_i(u)\| \neq 0$ and $\|Q_i(u)\| \neq 0, i = 1, 2, \forall u = 1, 2, \ldots, r$, then the the sequences $T_i(k)$, $P_i(k)$ and $Q_i(k), i = 1, 2$ generated by algorithm 6.1 satisfy the following statement, for $u, w = 1, 2, \ldots, r$ and $u \neq w$.

$$\sum_{i=1}^{2}\langle T_i(u), T_i(w)\rangle = 0, \quad \sum_{i=1}^{2}\langle Q_i(u), Q_i(w)\rangle = 0, \quad \sum_{i=1}^{2}\langle P_i(u), T_i(w)\rangle = 0. \quad (6.2.1)$$

Proof. Step 1: The conclusion will be proved by induction. Because of commutation of the real inner product, it is enough to prove three statements for $1 \leq u < w \leq r$.

For $u = 1, w = 2$, from algorithm 6.1, we have the following relation

$$\begin{aligned}
&\sum_{i=1}^{2}\langle T_i(1), T_i(2)\rangle\\
&\quad = \sum_{i=1}^{2}\langle T_i(1), T_i(1) - \delta(1)(L_{i1}^*(Q_i(1)) + L_{i2}^*(Q_i(1)))\rangle\\
&\quad = \sum_{i=1}^{2}\|T_i(1)\|^2 - \delta(1)\sum_{i=1}^{2}\langle L_{i1}(T_i(1)) + L_{i2}(T_i(1)), Q_i(1)\rangle\\
&\quad = \sum_{i=1}^{2}\|T_i(1)\|^2 - \delta(1)\sum_{i=1}^{2}\langle L_{i1}(P_i(1)) + L_{i2}(P_i(1)), P_i(1)\rangle\\
&\quad = \sum_{i=1}^{2}\|T_i(1)\|^2 - \delta(1)\sum_{i=1}^{2}\langle Q_i(1), Q_i(1)\rangle = 0,
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{2}\langle Q_i(1), Q_i(2)\rangle\\
&\quad=\sum_{i=1}^{2}\langle Q_i(1), L_{i1}(P_i(2))+L_{i2}(P_i(2))\rangle\\
&\quad=\sum_{i=1}^{2}\Big\langle Q_i(1), L_{i1}(T_i(2)+\sum_{i=1}^{2}\lambda(1)P_i(1))\\
&\qquad+L_{i2}(T_i(2)+\sum_{i=1}^{2}\lambda(1)P_i(1))\Big\rangle\\
&\quad=\sum_{i=1}^{2}\lambda(1)\|Q_i(1)\|^2+\sum_{i=1}^{2}\langle Q_i(1), L_{i1}(T_i(2))+L_{i2}(T_i(2))\rangle\\
&\quad=\sum_{i=1}^{2}\lambda(1)\|Q_i(1)\|^2-\frac{1}{\delta(1)}\sum_{i=1}^{2}\langle L_{i1}^*(R_i(2)-R_i(1))\\
&\qquad+L_{i2}^*(R_i(2)-R_i(1)), T_i(2)\rangle\\
&\quad=\sum_{i=1}^{2}\lambda(1)\|Q_i(1)\|^2-\frac{1}{\delta(1)}\sum_{i=1}^{2}\langle T_i(2)-T_i(1), T_i(2)\rangle=0,
\end{aligned}$$

and

$$\sum_{i=1}^{2}\langle P_i(1), T_i(2)\rangle=\sum_{i=1}^{2}\langle T_i(1), T_i(2)\rangle=0.$$

Step 2: Thus, for $u<w<r$ if we let

$$\sum_{i=1}^{2}\langle T_i(u), T_i(w)\rangle=0,\quad \sum_{i=1}^{2}\langle Q_i(u), Q_i(w)\rangle=0,\quad \sum_{i=1}^{2}\langle P_i(u), T_i(w)\rangle=0.$$

So we have

$$\begin{aligned}
&\sum_{i=1}^{2}\langle T_i(u), T_i(w+1)\rangle\\
&\quad=\sum_{i=1}^{2}\langle T_i(u), T_i(w)-\delta(w)(L_{i1}^*(Q_i(w))+L_{i2}^*(Q_i(w)))\rangle\\
&\quad=-\delta(w)\sum_{i=1}^{2}\langle T_i(u), L_{i1}^*(Q_i(w))+L_{i2}^*(Q_i(w))\rangle\\
&\quad=-\delta(w)\sum_{i=1}^{2}\langle L_{i1}(T_i(u))+L_{i2}(T_i(u)), Q_i(w)\rangle\\
&\quad=-\delta(w)\sum_{i=1}^{2}\langle L_{i1}(P_i(u)-\lambda(u-1)P_i(u-1))\\
&\qquad+L_{i2}(P_i(u)-\lambda(u-1)P_i(u-1)), Q_i(w)\rangle\\
&\quad=-\delta(w)\sum_{i=1}^{2}\langle Q_i(u)-\lambda(u-1)Q_i(u-1), Q_i(w)\rangle=0,
\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{2}\langle P_i(u),\, T_i(w+1)\rangle &= \sum_{i=1}^{2}\langle P_i(u),\, T_i(w)-\delta(w)(L_{i1}^*(Q_i(w))+L_{i2}^*(Q_i(w)))\rangle\\ &= -\delta(w)\sum_{i=1}^{2}\langle P_i(u), L_{i1}^*(Q_i(w))+L_{i2}^*(Q_i(w))\rangle\\ &= -\delta(w)\sum_{i=1}^{2}\langle L_{i1}(P_i(u))+L_{i2}(P_i(u)),\, Q_i(w)\rangle\\ &= -\delta(w)\sum_{i=1}^{2}\langle Q_i(u),\, Q_i(w)\rangle = 0.\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{2}\langle Q_i(u),\, Q_i(w+1)\rangle\\ &\quad= \sum_{i=1}^{2}\langle Q_i(u), L_{i1}(P_i(w+1))+L_{i2}(P_i(w+1))\rangle\\ &\quad= \sum_{i=1}^{2}\langle Q_i(u), L_{i1}(T_i(w+1)+\lambda(w)P_i(w))+L_{i2}(T_i(w+1)+\lambda(w)P_i(w))\rangle\\ &\quad= \sum_{i=1}^{2}\langle Q_i(u), L_{i1}(T_i(w+1))+L_{i2}(T_i(w+1))+\lambda(w)Q_i(w)\rangle\\ &\quad= -\frac{1}{\delta(u)}\sum_{i=1}^{2}\langle R_i(u+1)-R_i(u), L_{i1}(T_i(w+1))+L_{i2}(T_i(w+1))\rangle\\ &\quad= -\frac{1}{\delta(u)}\sum_{i=1}^{2}\langle L_{i1}^*(R_i(u+1)-R_i(u))+L_{i2}^*(R_i(u+1)-R_i(u)),\, T_i(w+1)\rangle\\ &\quad= -\frac{1}{\delta(u)}\sum_{i=1}^{2}\langle T_i(u+1)-T_i(u),\, T_i(w+1)\rangle = 0.\end{aligned}$$

For $u=w$, we also obtain the following relation

$$\begin{aligned}&\sum_{i=1}^{2}\langle T_i(w),\, T_i(w+1)\rangle\\ &\quad= \sum_{i=1}^{2}\langle T_i(w),\, T_i(w)-\delta(w)L_i^*(Q_i(w))\rangle\\ &\quad= \sum_{i=1}^{2}\|T_i(w)\|^2-\delta(w)\sum_{i=1}^{2}\langle T_i(w), L_i^*(Q_i(w))\rangle\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} \| T_i(w)\|^2 - \delta(w) \sum_{i=1}^{2} \langle L_i(T_i(w)), Q_i(w)\rangle \\
&= \sum_{i=1}^{2} \| T_i(w)\|^2 - \delta(w) \sum_{i=1}^{2} \langle L_i(P_i(w) - \lambda(w-1)P_i(w-1)), Q_i(w)\rangle \\
&= \sum_{i=1}^{2} \| T_i(w)\|^2 - \delta(w) \sum_{i=1}^{2} \langle Q_i(w) - \lambda(w-1)Q_i(w-1), Q_i(w)\rangle = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{2} \langle Q_i(w), Q_i(w+1)\rangle \\
&\quad = \sum_{i=1}^{2} \langle Q_i(w), L_{i1}(P_i(w+1)) + L_{i2}(P_i(w+1))\rangle \\
&\quad = \sum_{i=1}^{2} \langle Q_i(w), L_{i1}(T_i(w+1) + \lambda(w)P_i(w)) + L_{i2}(T_i(w+1) + \lambda(w)P_i(w))\rangle \\
&\quad = \sum_{i=1}^{2} \langle Q_i(w), L_{i1}(T_i(w+1) + L_{i2}(T_i(w+1) + \lambda(w)Q_i(w))\rangle \\
&\quad = \sum_{i=1}^{2} \lambda(w)\| Q_i(w)\|^2 - \frac{1}{\delta(w)} \sum_{i=1}^{2} \langle R_i(w+1) - R_i(w), L_{i1}(T_i(w+1)) + L_{i2}(T_i(w+1))\rangle \\
&\quad = \sum_{i=1}^{2} \lambda(w)\| Q_i(w)\|^2 - \frac{1}{\delta(w)} \sum_{i=1}^{2} \langle L_{i1}^{*}(R_i(w+1) - R_i(w)) \\
&\qquad + L_{i2}^{*}(R_i(w+1) - R_i(w)), T_i(w+1)\rangle \\
&\quad = \sum_{i=1}^{2} \lambda(w)\| Q_i(w)\|^2 - \frac{1}{\delta(w)} \sum_{i=1}^{2} \langle T_i(w+1) - T_i(w), T_i(w+1)\rangle \\
&\quad = \sum_{i=1}^{2} \lambda(w)\| Q_i(w)\|^2 - \frac{1}{\delta(w)} \sum_{i=1}^{2} \| T_i(w+1)\|^2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{2} \langle P_i(w), T_i(w+1)\rangle \\
&\quad = \sum_{i=1}^{2} \langle P_i(w), T_i(w) - \delta(w)(L_{i1}^{*}(Q_i(w)) + L_{i2}^{*}(Q_i(w)))\rangle \\
&\quad = \sum_{i=1}^{2} \langle T_i(w) + \lambda(w-1)T_i(w-1) + \lambda(w-1)\lambda(w-2)T_i(w-2) \\
&\qquad + \cdots + \lambda(w-1)\ldots\lambda(1)T_i(1), T_i(w)\rangle \\
&\qquad - \delta(w) \sum_{i=1}^{2} \langle P_i(w), L_{i1}^{*}(Q_i(w)) + L_{i2}^{*}(Q_i(w))\rangle \\
&\quad = \sum_{i=1}^{2} \| T_i(w)\|^2 - \delta(w) \sum_{i=1}^{2} \langle L_{i1}(P_i(w)) + L_{i2}(P_i(w)), Q_i(w)\rangle = 0.
\end{aligned}
$$

Thus, we complete the proof of the conclusion. □

Lemma 6.5. In algorithm 6.1, it is assumed that there exists a positive number l such that $\delta(l) = 0$ or $\delta(l) = \infty$, then $(X(l), Y(l))$ are the solutions of problem 6.1.

Proof. Now, it is assumed that $\delta(l) = 0$, then $\sum_{i=1}^{2} \| T_i(l)\|^2 = 0$. Let $\delta(l) = \infty$, then we obtain $\sum_{i=1}^{2} \| Q_i(l)\|^2 = 0$. This contradicts

$$\begin{aligned}&\sum_{i=1}^{2} \| T_i(l)\|^2 \\ &\quad = \sum_{i=1}^{2} \langle T_i(l) + \lambda(l-1) T_i(l-1) + \lambda(l-1)\lambda(l-2) T_i(l-2) \\ &\qquad + \cdots + \lambda(l-1)\ldots\lambda(1) T_i(1), T_i(l)\rangle \\ &\quad = \sum_{i=1}^{2} \langle P_i(l), T_i(l)\rangle = \sum_{i=1}^{2} \langle P_i(l), L_{i1}^*(R_i(l)) + L_{i2}^*(R_i(l))\rangle = 0.\end{aligned}$$

In this way, for $\delta(l) = 0$ or $\delta(l) = \infty$ we have

$$T_i(l) = L_{i1}^*(R_1(l)) + L_{i2}^*(R_2(l)) = 0, \quad i = 1, 2.$$

On this occasion, it follows from lemma 6.2 that $(X(l), Y(l))$ are the solutions of problem 6.1. Thus, we complete the proof of this lemma. □

Theorem 6.1. Supposed that coupled matrix equations (6.0.1) are consistent, then by using algorithm 6.1 the solution (X^*, Y^*) of problem 6.1 can be obtained for any arbitrary initial matrix pair $(X(l), Y(l))$.

Proof. For the sake of lemma 6.4, the sequence set generated by algorithm 6.1 $(T_1(j), T_2(j)), j = 1, 2, 3, \ldots, \sum_{i=1}^{2} 2m_i \times n_i$; is an orthogonal basis of the real inner product space $\mathfrak{C}^{m_1 \times n_1} \times \mathfrak{C}^{m_2 \times n_2}$ with dimension $\sum_{i=1}^{2} 2m_i \times n_i$. Under this circumstance, we have $\sum_{i=1}^{2} \left\| T_i(\sum_{i=1}^{2} 2m_i n_i + 1)\right\| = 0$. In this case, we can conclude that $(X(\sum_{i=1}^{2} 2m_i n_i + 1), Y(\sum_{i=1}^{2} 2m_i n_i + 1))$ are the solutions of problem 6.1 in the absence of roundoff errors. □

Theorem 6.2. If we choose the initial matrix as $L_{i1}^*(F(1)) + L_{i2}^*(F(1)),\ i = 1, 2$, in which $F(1) \in C^{m \times n}$ is an arbitrary complex matrix, or especially we choose the initial complex matrices $X(1) = 0$, $Y(1) = 0$, then the solution (X^*, Y^*) which are generated by algorithm 6.1 are the least norm solutions of problem 6.1.

Proof. If we choose the initial matrices of coupled matrix equations as $X(1) = L_{11}^*(F(1)) + L_{21}^*(F(1)$ and $Y(1) = L_{12}^*(F(1)) + L_{22}^*(F(1)$, then the matrix sequence generated by algorithm 6.1 $X(k), Y(k)$ can be stated as $X(k) = L_{11}^*(F(k)) + L_{21}^*(F(k))$ and $Y(k) = L_{12}^*(F(k)) + L_{22}^*(F(k))$ for a certain complex matrix $F(k) \in \mathfrak{C}^{m \times n}, k = 2, 3, \ldots$. Therefore, we can conclude that it must have a complex matrix $F^* \in \mathfrak{C}^{m \times n}$ such that $X^* = L_{11}^*(F^*) + L_{21}^*(F^*)$ and $Y^* = L_{12}^*(F^*) + L_{22}^*(F^*)$. Hence we assume $\tilde{X}^*, \tilde{Y}^*$ be an arbitrary solution of problem 6.1, then, according to lemma 6.3, there must be the complex matrix

$Z^* \in \mathfrak{C}^{m\times n}$ such that $\tilde{X}^* = X^* + Z^*$, $\tilde{Y}^* = Y^* + Z^*$ and $L_{11}(Z^*) + L_{21}(Z^*) = 0$, $L_{12}(Z^*) + L_{22}(\mathrm{Z}^*) = 0$. So we have the following relation

$$\langle X^*, Z^*\rangle = \langle L_{11}^*(F^*) + L_{21}^*(F^*), Z^*\rangle = \langle F^*, L_{11}(Z^*) + L_{21}(Z^*)\rangle = 0,$$

and

$$\langle Y^*, Z^*\rangle = \langle L_{12}^*(F^*) + L_{22}^*(F^*), Z^*\rangle = \langle F^*, L_{12}(Z^*) + L_{22}(Z^*)\rangle = 0.$$

Thus, we have

$$\|\tilde{X}^*\|^2 = \|X^* + Z^*\|^2 = \|X^*\|^2 + \|Z^*\|^2 + 2\langle X^*, Z^*\rangle = \|X^*\|^2 + \|Z^*\|^2 \geq \|X^*\|^2,$$

and

$$\|\tilde{Y}^*\|^2 = \|Y^* + Z^*\|^2 = \|Y^*\|^2 + \|Z^*\|^2 + 2\langle Y^*, Z^*\rangle = \|Y^*\|^2 + \|Z^*\|^2 \geq \|Y^*\|^2.$$

At this time, it is concluded that (X^*, Y^*) are the least Frobenius norm solutions of problem 6.1. So we complete the proof of the conclusion. □

Next, we will state as the minimization property of the proposed algorithm 6.1 in the following. This property shows that algorithm 6.1 converges smoothly.

Theorem 6.3. Supposed that $(X(1), Y(1)) \in \mathfrak{C}^{m\times n}$ be any initial matrices, then we have the following conclusion

$$\sum_{i=1}^{2} \|L_{i1}(X(k+1)) + L_{i2}(Y(k+1)) - E_i\| = \min_{X\in\psi_k} \sum_{i=1}^{2} \|L_{i1}(X) + L_{i2}(Y) - E_i, i = 1, 2\| \tag{6.2.2}$$

in which $(X(k+1), Y(k+1))$ are generated by algorithm 6.1 at the $k+1$-th iteration and ψ_k presents an affine subspace which has the following form

$$\psi_k = X(1) + Y(1) + \mathrm{span}((P_1(1), P_2(1)), (P_1(2), P_2(2)), \ldots, (P_1(k), P_2(k)). \tag{6.2.3}$$

Proof. According to equation (6.2.3), for any matrix $(X, Y) \in \psi_k$, there must exist numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$X = X(1) + \sum_{l=1}^{k} \alpha_l (P_1(l), P_2(l)),$$

and

$$Y = Y(1) + \sum_{l=1}^{k} \alpha_l (P_1(l), P_2(l)).$$

Now if we define a function f with respect to the variable $\alpha_1, \alpha_2, \ldots, \alpha_k$ as

$$f(\alpha_1, \alpha_2, \ldots, \alpha_k) = \sum_{i=1}^{2} \left\| L_{i1}\Big(X(1) + \sum_{l=1}^{k} \alpha_l P_i(l)) + L_{i2}(Y(1) + \sum_{l=1}^{k} \alpha_l P_i(l)\Big) - E_i \right\|^2,$$

in which the function f is continuous and differentiable. So from lemma 6.4 we have

$$\begin{aligned}
&f(\alpha_1,\alpha_2,\ldots,\alpha_k)\\
&\quad=\sum_{i=1}^{2}\left\|L_{i1}(X(1))+L_{i2}(Y(1))-E_i+\sum_{l=1}^{k}\alpha_l(L_{i1}(P_i(l))+L_{i2}(P_i(l)))\right\|^2\\
&\quad=\sum_{i=1}^{2}\|R_i(1)\|^2+\sum_{i=1}^{2}\sum_{l=1}^{k}\alpha_l^2\|Q_i(1)\|^2-2\sum_{i=1}^{2}\sum_{l=1}^{k}\alpha_l\langle Q_i(1),R_i(1)\rangle.
\end{aligned}$$

At the same time, if we now consider the problem of minimizing the function $f(\alpha_1,\alpha_2,\ldots,\alpha_k)$, we can obtain

$$\begin{aligned}
&\min_{\alpha_l} f(\alpha_1,\alpha_2,\ldots,\alpha_k)\\
&\quad=\min_{X,Y\in\psi_k}\sum_{i=1}^{2}\|L_{i1}(X)+L_{i2}(Y)-E_i\|^2,\quad l=1,2,\ldots,k.
\end{aligned}$$

From this function, the minimum occurs when

$$\frac{\partial(\alpha_1,\alpha_2,\ldots,\alpha_k)}{\partial\alpha_l}=0\quad\text{for}\quad l=1,2,\ldots,k.$$

Consequently, we can get

$$\alpha_l=\frac{\sum_{i=1}^{2}\langle Q_i(l),R_{i1}(l)\rangle}{\sum_{i=1}^{2}\|Q_i(l)\|^2}.$$

Since algorithm 6.1, we have

$$R_i(1)=R_i(l)+\delta(l-1)Q_i(l-1)+\delta(l-2)Q_i(l-2)+\cdots+\delta(1)Q_i(1).$$

Therefore, according to lemma 6.4 we have

$$\begin{aligned}
\alpha_l&=\frac{\sum_{i=1}^{2}\langle Q_i(l),R_i(l)\rangle}{\sum_{i=1}^{2}\|Q_i(l)\|^2}=\frac{\sum_{i=1}^{2}\langle P_i(l),L_{i1}^*(R_i(l))+L_{i2}^*(R_i(l))\rangle}{\sum_{i=1}^{2}\|Q_i(l)\|^2}\\
&=\frac{\sum_{i=1}^{2}\langle P_i(l),S_i(l)\rangle}{\sum_{i=1}^{2}\|Q_i(l)\|^2}=\frac{\sum_{i=1}^{2}\langle S_i(l)+\lambda(l-1)P_i(l-1),S_i(l)\rangle}{\sum_{i=1}^{2}\|Q_i(l)\|^2}\\
&=\frac{\sum_{i=1}^{2}\|S_i(l)\|^2}{\sum_{i=1}^{2}\|Q_i(l)\|^2}=\delta(l).
\end{aligned}$$

In a similar way, we can prove the case of $i=2$. By this means, we complete the proof. □

By theorem 6.1, for any initial matrix, we can conclude that the residual norm of the solution generalized by algorithm 6.1 minimizes at the $k+1-$ th iteration in the affine subspace ψ_k. So we have

$$\sum_{i=1}^{2} \|L_{i1}(X(k+l)) + L_{i2}(Y(k+l)) - E_i\|^2 \leq \sum_{i=1}^{2} \|L_i(X(k)) + L_{i2}(Y(k)) - E_i\|^2, \tag{6.2.4}$$

This indicates that the residual norm $\sum_{i=1}^{2} \|R_i(1)\|, \sum_{i=1}^{2} \|R_i(2)\|, \ldots, i = 1, 2$, is monotonically decreasing. This shows that algorithm 6.1 can process fast and smoothly convergence.

Now problem 6.2 will be considered. For a given matrix X_0, from problem 6.1 we have

$$\begin{aligned} &\min_{X, Y \in C^{m\times n}} \sum_{i=1}^{2} \|L_{i1}(X) + L_{i2}(Y) - E_i\| \\ &= \min_{X, Y \in C^{m\times n}} \sum_{i=1}^{2} \|L_{i1}(X - X_0) + L_{i2}(Y - Y_0) - (E_i - L_{i1}(X_0) - L_{i2}(Y_0))\|. \end{aligned} \tag{6.2.5}$$

If we let the set $\grave{E}_i = E_i - L_{i1}(X_0) - L_{i2}(Y_0)$ and $\grave{X} = X - X_0$, $\grave{Y} = Y - Y_0$, then problem 6.2 is equivalent to find the least Frobenius norm solution (X_1^*, Y_1^*) of

$$\min_{\grave{X}, \grave{Y} \in \mathfrak{C}^{m\times n}} \sum_{i=1}^{2} \left\| L_{i1}(\grave{X}) + L_{i2}(\grave{Y}) - \grave{E}_i \right\|, \tag{6.2.6}$$

with the initial matrix $\grave{X}(1) = L_{11}^*(F) + L_{21}^*(F)$, $\grave{Y}(1) = L_{12}^*(F) + L_{22}^*(F)$, in which $F \in C^{m\times n}$ is an arbitrary matrix, or especially $\grave{X}(1) = 0$, $\grave{Y}(1) = 0$. Therefore, we can state the solutions $\|\check{X} - X_0\| = \min_{X\in G_r} \|X - X_0\|$ and

$$\|\check{Y} - Y_0\| = \min_{Y\in G_r} \|Y - Y_0\|,$$

of problem 6.2 as

$$\check{X} = \grave{X}^* + X_0, \tag{6.2.7}$$

and

$$\check{Y} = \grave{Y}^* + Y_0. \tag{6.2.8}$$

Thus, it is concluded that the solution of problem 6.2 can be obtained through the solution of problem 6.1.

6.3 Numerical Examples

In this section, we give two numerical examples to check our proposed CGLS-M iterative algorithm. The first example proves our proposed iterative algorithm is better than the existing LSQR iterative algorithm. The second example shows that our presented iterative algorithm is also effective for any random coefficient matrices.

Example 6.1. Solve the least Frobenius norm solution of the following coupled matrix equations

$$\begin{cases} A_1XB_1 + C_1\overline{X}D_1 + E_1X^HF_1 = M_1, \\ A_2\overline{X}B_2 + C_2X^TD_2 = M_2, \end{cases} \tag{6.3.1}$$

in which the coefficient matrices are

$$A_1 = \begin{pmatrix} 12+i & 1-3i & 6 & 4+3i \\ 2+i & 10 & 2 & 0 \\ 5-6i & 2+13i & -2i & i \\ 1 & 12 & 10 & 9i \end{pmatrix}, B_1 = \begin{pmatrix} 3 & -2i & 4+6i \\ 9+8i & 4-6i & 11 \\ 2-i & -9i & 12 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} i & 10 & -2i & 4 \\ 1 & 13i & 1-2i & 1 \\ 1-8i & 11-9i & 21 & i \\ 9-8i & 1 & i & -8i \end{pmatrix}, D_1 = \begin{pmatrix} -i & i & 6+8i \\ 4i & 19 & 0 \\ 23 & 26+i & 9i \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 3 & 15 & 1 & 2+i \\ 11 & 1+2i & 11 & \\ 18 & 21 & i & \\ 8i & 21 & i & 8-i \end{pmatrix}, D_2 = \begin{pmatrix} 9-i & 15 & 2+i \\ 12+3i & 1+2i & 11 \\ 1+9i & 21 & i \\ 21 & -i & 8-i \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -i & 8 & 1 & 2i \\ 11 & 21-3i & 12 & 11 \\ 78 & 11 & 21 & i \\ 9-8i & 11 & i & -8i \end{pmatrix}, B_2 = \begin{pmatrix} 9 & 1-5i & 2i \\ 11 & 2+3i & 12 \\ 18 & -9i & 0 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 13 & 5 & i \\ 7 & 1+23i & 1-i \\ 30-i & 21 & 8i \\ 8-i & 2-6i & 8-i \end{pmatrix}, F_1 = \begin{pmatrix} 13 & 5 & 2+6i \\ 1 & 0 & 11 \\ 9 & 21 & i \\ 8-i & 21 & -i \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 9323+4757i & 11447+1515i & 4774+1172i \\ 543+11632i & 885+10010i & -423+2254i \\ 17415+7013i & 22836+3936i & 4649+4863i \\ 5119-1418i & 9271-8442i & 6220-289i \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 8953+2835i & 8019-249i & 4094+339i \\ 19013+4942i & 6635-3474i & 8742-389i \\ 33450+6034i & 18057+3909i & 26325+1270i \\ 15116-2658i & 9423+343i & 8147-2956i \end{pmatrix}.$$

By some computation, we can obtain that the coupled matrix equation (6.3.1) are consistent and have the solution

$$X = \begin{pmatrix} 3i & 21+i & 0 \\ 6i & 9 & 8-9i \\ 11-i & 4 & 2-9i \\ 2-3i & 4 & 12 \end{pmatrix}.$$

Here, if the initial matrix $X_1 = \text{ones}(4,3)$ can be chosen, applying algorithm 6.1 the iterative least square solution can be solved as

$$X(243) = \begin{pmatrix} 2.9999i & 21.0000+0.9999i & 0.0000 \\ 5.9999i & 9.0000 & 8.0000-8.9999i \\ 11.0000-0.9999i & 3.9999 & 1.9999-9.0000i \\ 1.9999-2.9999i & 3.9999 & 11.9999 \end{pmatrix}.$$

The residual and relative error of coupled matrix equation (6.3.1) are showed in figures 6.3.1 and 6.3.2, in which

$$r_k = \log_{10}\sqrt[2]{||M_1 - G||^2 + ||M_2 - H||^2},$$

where $G = A_1X(k)B_1 + C_1\overline{X(k)}D_1 + E_1X(k)^H F_1$, $H = A_2\overline{X(k)}B_2 + C_2X(k)^T D_2$ and

$$\text{err}(k) = \frac{\|X(k) - X\|}{\|X\|}.$$

Remark 1. It is followed from figures 6.3.1 and 6.3.2 that the residual and the relative error of the solution to coupled matrix equation (6.3.1) are reduced with the addition of an iterative number. By this result, we can see the iterative solution of coupled matrix equations converges to the exact solution. Furthermore, with the addition of an iterative number, it is also shown that the residual and the relative error of the LSQR-M iterative algorithm of coupled matrix equations are not varied. At the same time, it is proved that the residual and the relative error of our iterative algorithm are smaller than those of the LSQR-M iterative algorithm. Therefore, our proposed method (MCGLS iterative algorithm) is an effective and great algorithm.

Example 6.2. Solve the least Frobenius norm solution of the following coupled matrix equations

$$\begin{cases} A_1XB_1 + C_1\overline{X}D_1 + E_1YF_1 = M_1, \\ A_2XB_2 + C_2\overline{X}D_2 + E_2Y\mathrm{F}_2 = M_2, \end{cases} \tag{6.3.2}$$

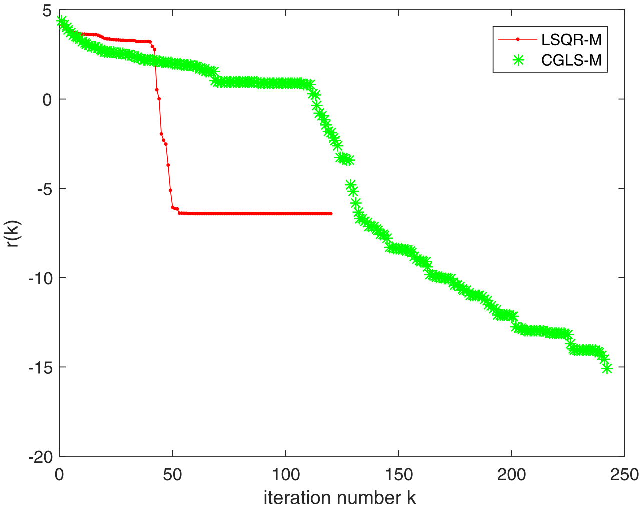

FIG. 6.3.1 – The residual of solution for example 6.1.

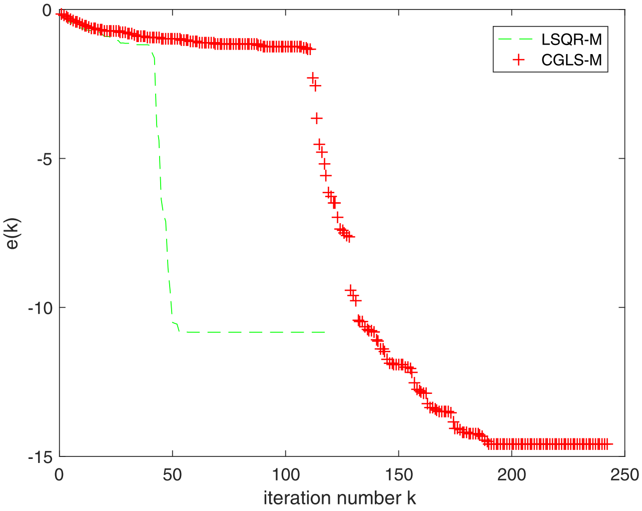

FIG. 6.3.2 – The relative error of solution for example 6.1.

where

$$A_1 = \mathrm{rand}(6,7) + i * \mathrm{rand}(6,7), \quad B_1 = \mathrm{rand}(5,4) + i * \mathrm{rand}(5,4),$$

$$C_1 = \mathrm{rand}(6,7) + i * \mathrm{rand}(6,7), \quad D_1 = \mathrm{rand}(5,4) + i * \mathrm{rand}(5,4),$$

$$A_2 = \text{rand}(6,7) + i * \text{rand}(6,7), \quad B_2 = \text{rand}(5,4) + i * \text{rand}(5,4),$$

$$C_2 = \text{rand}(6,7) + i * \text{rand}(6,7), \quad D_2 = \text{rand}(5,4) + i * \text{rand}(5,4),$$

$$E_2 = \text{rand}(6,7) + 1i * \text{rand}(6,7), \quad F_2 = \text{rand}(5,4) + i * \text{rand}(5,4),$$

$$E_1 = \text{rand}(6,7) + 1i * \text{rand}(6,7), \quad F_1 = \text{rand}(5,4) + i * \text{rand}(5,4),$$

$$M_1 = \begin{pmatrix} -6.6343 + 29.7004i & -11.6641 + 26.6410i & 4.7953 + 26.2878i & -3.4258 + 26.0280i \\ -4.3583 + 20.2072i & -9.0990 + 16.9895i & -2.7453 + 17.6718i & -0.3306 + 16.9502i \\ -9.0665 + 24.7064i & -13.0560 + 22.1559i & -7.2720 + 21.8593i & -6.8525 + 21.3105i \\ -8.0747 + 19.2879i & -10.7742 + 17.5695i & -5.4971 + 17.6245i & -5.7757 + 15.8973i \\ -8.7840 + 20.3774i & -12.7093 + 17.5514i & -6.5985 + 18.3941i & -6.6394 + 18.9002i \\ -5.7835 + 23.3203i & -10.7752 + 22.2704i & -4.3001 + 21.0105i & -4.1352 + 19.7401i \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -14.5118 + 26.6063i & -11.6376 + 32.5976i & -10.4847 + 21.0546i & -11.7599 + 27.9033i \\ -11.5972 + 28.1350i & -5.4724 + 34.2821i & -7.6472 + 22.9915i & -7.8647 + 28.8984i \\ -15.8140 + 18.5914i & -11.6417 + 26.0488i & -12.4484 + 15.4641i & -14.1854 + 23.0876i \\ -14.1510 + 21.0841i & -8.8672 + 27.9473i & -10.8980 + 18.4100i & -11.0275 + 25.8017i \\ -13.5717 + 21.0144i & -9.6517 + 27.6175 & -10.3281 + 16.0999i & -10.1937 + 22.7480i \\ -17.3981 + 23.2946i & -14.0598 + 31.9706i & -13.7004 + 19.7497i & -15.4352 + 26.8111i \end{pmatrix}.$$

By some computation, we can obtain that the coupled matrix equations (6.3.2) are consistent and have the solution

$$X = \left(\begin{matrix} 0.4417+0.2880i & 0.1017+0.6079i & 0.5054+0.7990i \\ 0.0132+0.6925i & 0.9954+0.7413i & 0.7614+0.7343i \\ 0.8972+0.5567i & 0.3321+0.1048i & 0.6311+0.0513i \\ 0.1967+0.3965i & 0.2973+0.1279i & 0.0899+0.0729i \\ 0.0934+0.0616i & 0.0620+0.5495i & 0.0809+0.0885i \\ 0.3074+0.7802i & 0.2982+0.4852i & 0.7772+0.7984i \\ 0.4561+0.3376i & 0.0464+0.8905i & 0.9051+0.9430i \end{matrix}\right.$$

$$\left.\begin{matrix} 0.5338+0.6837i & 0.0485+0.6538i \\ 0.1092+0.1321i & 0.6679+0.7491i \\ 0.8258+0.7227i & 0.6035+0.5832i \\ 0.3381+0.1104i & 0.5261+0.7400i \\ 0.2940+0.1175\mathrm{i} & 0.7297+0.2348i \\ 0.7463+0.6407i & 0.7073+0.7350i \\ 0.0103+0.3288i & 0.7814+0.9706i \end{matrix}\right),$$

and

$$Y = \left(\begin{matrix} 0.8669+0.6555i & 0.3677+0.7690i & 0.2290+0.6099i \\ 0.0862+0.1098i & 0.2060+0.3960i & 0.6419+0.0594i \\ 0.3664+0.9338i & 0.0867+0.2729i & 0.4845+0.3158i \\ 0.3692+0.1875i & 0.7719+0.0372i & 0.1518+0.7727i \\ 0.6850+0.2662i & 0.2057+0.6733i & 0.7819+0.6964i \\ 0.5979+0.7978i & 0.3883+0.4296i & 0.1006+0.1253i \\ 0.7894+0.4876i & 0.5518+0.4517i & 0.2941+0.1302i \end{matrix}\right.$$

$$\left.\begin{matrix} 0.2374+0.0924i & 0.2916+0.6318i \\ 0.5309+0.0078i & 0.6035+0.1265i \\ 0.0915+0.4231i & 0.9644+0.1343i \\ 0.4053+0.6556i & 0.4325+0.0986i \\ 0.1048+0.7229i & 0.6948+0.1420i \\ 0.1123+0.5312i & 0.7581+0.1683i \\ 0.7844+0.1088i & 0.4326+0.1962i \end{matrix}\right).$$

According to algorithm 6.1, if we let the initial matrix $X(1) = \text{ones}(7,5)$, $Y(1) = \text{ones}(7,5)$, the iterative solution converge to the exact solution and we can obtain figure 6.3.3. The corresponding residual is

$$r(324) = 9.5931 \times 10^{-16},$$

where

$$r_k = \log_{10}\sqrt[2]{\|M_1 - M_1'\|^2 + \|M_2 - M_2'\|^2},$$

$$M_1' = A_1X(k)B_1 - C_1\overline{X(k)}D_1 - E_1Y(k)F_1,$$

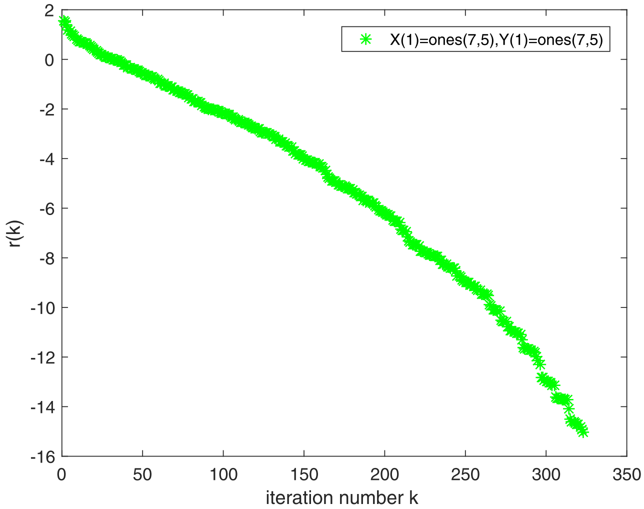

Fig. 6.3.3 – The residual of solution for example 6.2.

and

$$M_2' = A_2X(k)B_2 - C_2\overline{X(k)}D_2 - E_2Y(k)F_2.$$

Remark 2. From figure 6.3.3, for any random coefficient matrices, it's shown that the residual is becoming smaller and smaller with the addition of an iterative number, and the iterative solution converges to the exact solution. In addition, it is also proved that our proposed MCGLS iterative algorithm is a magnificent algorithm.

6.4 Conclusions

In this chapter, the MCGLS iterative algorithm is constructed for solving the least Frobenius norm solution to a coupled matrix equations. By use of this proposed iterative algorithm, on the one hand, for any initial matrix pair $(X(1), Y(1))$, the least Frobenius norm solution can be obtained. On the other hand, if a matrix pair (X_0, Y_0) are given, the optimal approximation solution $(\check{X}, \check{Y})$ can be derived by first finding the least Frobenius norm solution of a new corresponding coupled matrix equations. It is worth noting that our proposed iterative algorithm is suitable for a family of coupled matrix equations whose coefficient matrices are random matrices. In addition, the numerical example shows that our proposed MCGLS iterative algorithm is better than the existing LSQR iterative algorithm.

Chapter 7

Explicit Solutions to the Matrix Equation $X - AXB = CY + R$

In this chapter, we are concerned with the explicit solutions to the nonhomogeneous Yakubovich matrix equation

$$X - AXB = CY + R, \tag{7.0.1}$$

where $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{p\times p}$ and $C \in \mathfrak{R}^{n\times r}$ are given real matrices, $X \in \mathfrak{R}^{n\times p}$ and $Y \in \mathfrak{R}^{r\times p}$ are the matrices to be determined. Obviously, the well known Yakubovich matrix equation (7.0.2) and generalized Sylvester matrix equation (7.0.3)

$$X - AXB = CY, \tag{7.0.2}$$

and

$$AX - EXF = CY, \tag{7.0.3}$$

are the special cases of the nonhomogeneous Yakubovich matrix equation.

7.1 Solutions to the Real Matrix Equation $X - AXB = CY + R$

In this section, we provide the closed-form solutions of the nonhomogeneous Yakubivich matrix equation (7.0.1) by Smith normal form reduction and polynomial matrix. In order to obtain the result, we need the following lemma.

Lemma 7.1. Given matrices $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{p\times p}$ and $C \in \mathfrak{R}^{n\times r}$, suppose that $\{s|\det(I_n - sA) = 0\} \bigcap \lambda(B) = \phi$. Then there exist polynomial matrices $L(s) \in \mathfrak{R}^{n\times n}[s]$, $H(s) \in \mathfrak{R}^{r\times n}[s]$, and polynomial $\Delta(s) \in R[s]$ such that

$$(I_n - sA)L(s) - CH(s) = \Delta(s)I_n, \tag{7.1.1}$$

in which $\Delta(s) \neq 0$, for any $s \in \lambda(B)$.

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Proof. Assume $P(s) \in \mathfrak{R}^{n\times n}[s]$ and $Q(s) \in \mathfrak{R}^{(n+r)\times(n+r)}[s]$ be two uniocular polynomial matrices. Now we transform $[I_n - sA \ \ -C]$ into its Smith normal form

$$P(s)[I_n - sA \ \ -C]\,Q(s) = [\mathrm{diag}_{i=1}^{n} d_i(s) \ 0], \tag{7.1.2}$$

where $d_{i-1}(s)|d_i(s) \in R[s], i \in I[2,n]$. Since the relation $\{s|\det(I_n - sA) = 0\} \bigcap \lambda(B) = \phi$ holds, then $d_i(\gamma) \neq 0, i \in I[1,n]$, for any $\gamma \in \lambda(B)$. Partition $Q(s)$ into $[Q_1(s), Q_2(s)]$ with $Q_1(s) \in R^{(n+r)\times n}[s]$. It follows from (7.1.2) that

$$\begin{aligned} &P(s)[I_n - sA \ \ -C]\,Q_1(s) = \mathrm{diag}_{i=1}^{n} d_i(s) \\ &\Longleftrightarrow P(s)[I_n - sA \ \ -C]\,Q_1(s)\mathrm{diag}_{i=1}^{n}\frac{d_n(s)}{d_i(s)} = I_n d_n(s) \\ &\Longleftrightarrow [I_n - sA \ \ -C]\left(Q_1(s)\mathrm{diag}_{i=1}^{n}\frac{d_n(s)}{d_i(s)}\right)P(s) = I_n d_n(s). \end{aligned}$$

Thus, one can choose $\Delta(s) = d_n(s)$ and

$$\begin{bmatrix} L(s) \\ H(s) \end{bmatrix} = \left(Q_1(s)\mathrm{diag}_{i=1}^{n}\frac{d_n(s)}{d_i(s)}\right)P(s),$$

to satisfy the relation (7.1.1). □

In the following, we are going to provide a closed-form solution of the nonhomogeneous Yakubovich matrix equation (7.0.1) by the above lemma.

Theorem 7.1. Given matrices $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{p\times p}$, $C \in R^{n\times r}$ and $R \in \mathfrak{R}^{n\times p}$, suppose that polynomial matrices $L(s) = \sum_{i=0}^{t-1} L_i s^i \in \mathfrak{R}^{n\times n}[s]$ and $H(s) = \sum_{i=0}^{t} H_i s^i \in \mathfrak{R}^{r\times n}[s]$, and polynomial $\Delta(s) = \sum_{i=0}^{t}\Delta_i s^i \in R[s]$ satisfy (7.1.1) with $\Delta(s) \neq 0$, for any $s \in \lambda(B)$. Further, let the relation $\{s|\det(I_n - sA) = 0\} \bigcap \lambda(B) = \phi$ hold, and

$$\begin{aligned} &f_{(I_n,A)}(s) = \det(I_n - sA) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0, \alpha_0 = 1, \\ &\mathrm{adj}(I_n - sA) = R_{n-1}s^{n-1} + \cdots + R_1 s + R_0. \end{aligned}$$

Then all the solutions to the nonhomogeneous Yakubovich matrix equation (7.0.1) can be stated as

$$\begin{cases} X = \sum\limits_{i=0}^{n-1} R_i CZB^i + \sum\limits_{i=0}^{t-1} L_i R[\Delta(B)]^{-1}B^i, \\ Y = Zf_{(I_n,A)}(B) + \sum\limits_{i=0}^{t} H_i R[\Delta(B)]^{-1}B^i. \end{cases} \tag{7.1.3}$$

where $Z \in \mathfrak{R}^{r\times p}$ is an arbitrary chosen free parameter matrix.

Proof. We first show that the matrices X and Y given in (7.1.3) are solutions of the matrix equation (7.0.1). A direct calculation gives

$$\begin{aligned}
&X - AXB\\
&\quad= \sum_{i=0}^{n-1} R_i CZB^i + \sum_{i=0}^{t-1} L_i R[\Delta(B)]^{-1} B^i - A\sum_{i=0}^{n-1} R_i CZB^i B - A\sum_{i=0}^{t-1} L_i R[\Delta(B)]^{-1} B^i B\\
&\quad= \sum_{i=0}^{n-1} R_i CZB^i - A\sum_{i=0}^{n-1} R_i CZB^{i+1} + \sum_{i=0}^{t-1} L_i R[\Delta(B)]^{-1} B^i - A\sum_{i=0}^{t-1} L_i R[\Delta(B)]^{-1} B^{i+1}\\
&\quad= R_0 CZ + \sum_{i=1}^{n-1} (R_i - AR_{i-1}) CZB^i - AR_{n-1} CZB^n + L_0 R[\Delta(B)]^{-1}\\
&\qquad + \sum_{i=1}^{t-1} (L_i - AL_{i-1}) R[\Delta(B)]^{-1} B^i - AL_{t-1} R[\Delta(B)]^{-1} B^t.
\end{aligned} \tag{7.1.4}$$

According to $(I_n - sA)\mathrm{adj}(I_n - sA) = I_n \det(I_n - sA)$, it is easy to obtain

$$\begin{cases} R_0 = \alpha_0 I_n, \\ R_i - AR_{i-1} = \alpha_i I_n, \quad i \in I[1, n-1], \\ -AR_{n-1} = \alpha_n I_n. \end{cases} \tag{7.1.5}$$

So we have

$$R_0 CZ + \sum_{i=1}^{n-1} (R_i - AR_{i-1}) CZB^i - AR_{n-1} CZB^n = CZ \sum_{i=0}^{n} \alpha_i B^i = CZf_{(I_n, A)}(B). \tag{7.1.6}$$

Moreover, comparing the power of s in (7.1.1) yields

$$\begin{cases} L_0 = CH_0 + \Delta_0 I_n, \\ L_i - AL_{i-1} = CH_i + \Delta_i I_n, \quad i \in I[1, t-1], \\ -AL_{t-1} = CH_t + \Delta_t I_n. \end{cases} \tag{7.1.7}$$

With these relations, we can derive

$$\begin{aligned}
&L_0 R[\Delta(B)]^{-1} + \sum_{i=1}^{t-1} (L_i - AL_{i-1}) R[\Delta(B)]^{-1} B^i - AL_{t-1} R[\Delta(B)]^{-1} B^t\\
&\quad= (CH_0 + \Delta_0 I_n) R[\Delta(B)]^{-1} + \sum_{i=1}^{t-1} (CH_i + \Delta_i I_n) R[\Delta(B)]^{-1} B^i\\
&\qquad + (CH_t + \Delta_t I_n) R[\Delta(B)]^{-1} B^t\\
&\quad= \sum_{i=0}^{t} CH_i R[\Delta(B)]^{-1} B^i + \sum_{i=0}^{t} \Delta_i R[\Delta(B)]^{-1} B^i\\
&\quad= \sum_{i=0}^{t} CH_i R[\Delta(B)]^{-1} B^i + R[\Delta(B)]^{-1} \sum_{i=0}^{t} \Delta_i B^i\\
&\quad= \sum_{i=0}^{t} CH_i R[\Delta(B)]^{-1} B^i + R[\Delta(B)]^{-1} \Delta(B)\\
&\quad= \sum_{i=0}^{t} CH_i R[\Delta(B)]^{-1} B^i + R.
\end{aligned} \tag{7.1.8}$$

Substituting (7.1.6) and (7.1.8) into (7.1.4), we have

$$\begin{aligned} X - AXB &= CZf_{(I_n,A)}(B) + \sum_{i=0}^{t} CH_iR[\Delta(B)]^{-1}B^i + R \\ &= C\left(Zf_{(I_n,A)}(B) + \sum_{i=0}^{t} H_iR[\Delta(B)]^{-1}B^i \right) + R \\ &= CY + R. \end{aligned}$$

Therefore, the matrices X and Y given in (7.1.3) satisfy the matrix equation (7.0.1).

Next, the completeness of solution (7.1.3) will be proved. In relation (7.1.3), the free parameter matrix Z is included, which is an $r \times p$ matrix. So the parametric matrix Z has rp parameters. In this case, solution (7.1.3) has rp parameters. Putting $\mathrm{vec}(\cdot)$ on both sides of expression (X, Y) in (7.1.3), we have

$$\begin{bmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{bmatrix} = \mathbf{\Psi} \begin{bmatrix} \mathrm{vec}(Z) \\ \mathrm{vec}(R) \end{bmatrix},$$

in which $\mathbf{\Psi} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, $M_{11} = \sum_{i=1}^{n-1} [(B^i)^T \otimes (R_iC)]$,

$$M_{12} = \sum_{i=0}^{t-1} \left[((\Delta(B))^{-1}B^i)^T \otimes (L_i) \right],\ M_{21} = [f_{(I_n,A)}(B)]^T \otimes I_r,$$

$$M_{22} = \sum_{i=0}^{t} \left[((\Delta(B))^{-1}B^i)^T \otimes (H_i) \right].$$

Denote the set

$$\widetilde{\mathcal{R}} = \left\{ \begin{bmatrix} X \\ Y \end{bmatrix} \middle| \begin{bmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{bmatrix} = \mathbf{\Psi} \begin{bmatrix} \mathrm{vec}(Z) \\ \mathrm{vec}(R) \end{bmatrix}, Z \in \Re^{r \times p} \right\},$$

and according to the above section of the proof in theorem 7.1, we have $\widetilde{\mathcal{R}} \subset \mathcal{R}$. Obviously, $\widetilde{\mathcal{R}} = \mathcal{R}$ ($\mathcal{R}$ denote the solution set of matrix equation (7.0.1)) if and only if $\dim(\widetilde{\mathcal{R}}) = rp$. Since $\dim(Z) = rp$, $\dim(\widetilde{\mathcal{R}}) = rp$ if and only if $\mathrm{rank}(\mathbf{\Psi}) = rp$. Obviously, $\mathrm{rank}(\mathbf{\Psi}) = rp$ if M_{21} is nonsingular. M_{21} is nonsingular if and only if $f_{(I_n,A)}(B)$ is a nonsingular matrix.

If we assume $f_{(I_n,A)}(s) = \det(I_n - sA) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0, \alpha_0 = 1$, and $\mathrm{adj}(I_n - sA) = R_{n-1}s^{n-1} + \cdots + R_1 s + R_0$, because of $\{s | \det(I_n - sA) = 0\} \bigcap \lambda(B) = \phi$, the matrix $f_{(I_n,A)}(B) = \alpha_n B^n + \cdots + \alpha_1 B + \alpha_0 I_p$ is a nonsingular matrix. Therefore, the mapping $Z \rightarrow (X, Y)$ defined by (7.1.3) is injective. It is showed the completeness of solution (7.1.3). Thus the proof has been completed. $\square$

Remark 1. In[142], in order to obtain the solution, it demands the matrices $D_i, i = 1, 2, \ldots, \psi$. satisfying the unknown matrices $X = BD_0 + ABD_1 + \cdots + A^{\psi-1}BD_{\psi-1} + A^{\psi}BD_{\psi}$ and $XY = R$. In order to make the matrices $D_i, i = 1, 2, \ldots, \psi$. easier, it expect that the matrices X and Y have low dimension. In this case, (X, Y) is a pair of full rank factorization of R. But in the present chapter, it has no this limit.

From theorem 7.1, it is easy to know that under the condition of $\{s|\det(I_n - sA) = 0\} \bigcap \lambda(B) = \phi$ all the solutions of the matrix equation (7.0.2) can be explicitly given by

$$\begin{cases} X = \sum\limits_{i=0}^{n-1} R_i CZB^i, \\ Y = Zf_{(I_n, A)}(B). \end{cases} \tag{7.1.9}$$

Moreover, by theorem 7.1 it is obvious that a special solution of the matrix equation (7.0.1) is

$$\begin{cases} X = \sum\limits_{i=0}^{t-1} L_i R[\Delta(B)]^{-1} B^i, \\ Y = \sum\limits_{i=0}^{t} H_i R[\Delta(B)]^{-1} B^i. \end{cases}$$

Regarding the solution of $\alpha_i, i \in I[0, n]$, and $R_i, i \in I[0, n-1]$, the so-called generalized Leverrier algorithm[139] can be stated as the following relation:

$$\begin{cases} R_i = AR_{i-1} + \alpha_i I_n, i \in I[1, n-1], \quad R_0 = I, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i \in I[1, n]. \\ \alpha_i = \frac{\text{trace}(AR_{i-1})}{i}. \end{cases} \tag{7.1.10}$$

For relative lower-order cases, the Smith norm form reduction can be realized by applying elementary polynomial matrix transformation. For high-order cases, the Maple function "smith" can be readily employed.

Similarly to the reference[145], we have the following equivalent form of the presented solution (7.1.3). For this solution, a lemma is necessary. The idea of the proof of this lemma is motivated by[145,148].

Lemma 7.2. Suppose $L_i, i \in I[0, t-1]$, and $H_i, \Delta_i, i \in I[0, t]$, be the coefficients of $L(s)$, $H(s)$ and $\Delta(s)$, respectively. Then the relation (7.1.1) holds if and only if $H_i, \Delta_i, i \in I[0, t]$, satisfy

$$\sum_{i=0}^{t} (A)^{t-i} CH_i = -\sum_{i=0}^{t} \Delta_i (A)^{t-i}, \tag{7.1.11}$$

and $L_i, i \in I[0, t-1]$, satisfy

$$[L_0 \ L_1 \cdots L_{t-1}] = Ctr_t(A, C)S_t(H(s)) + Ctr_t(A, I_n)S_t(\Delta(s)I_n). \tag{7.1.12}$$

Proof. By theorem 7.1 we can know that $L_i, i \in I[0, t-1]$, and $H_i, \Delta_i, i \in I[0, t]$ satisfy (7.1.1) if and only if they satisfy (7.1.7). Therefore, we only need to show that the relation (7.1.7) is equivalent to the relations (7.1.11) and (7.1.12).

From the recursive relation (7.1.7), a direct calculation gives

$$\begin{cases} L_0 = CH_0 + \Delta_0 I_n, \\ L_1 = CH_1 + ACH_0 + \Delta_1 I_n + A\Delta_0 I_n, \\ \cdots \\ L_{t-1} = CH_{t-1} + ACH_{t-2} + \cdots + A^{t-1}CH_0 \\ \qquad\qquad + \Delta_{t-1} I_n + \Delta_{t-2} A + \cdots + \Delta_0 A^{t-1}. \end{cases} \tag{7.1.13}$$

and

$$0 = CH_t + ACH_{t-1} + \cdots + A^t CH_0 + \Delta_t I + \Delta_{t-1} A + \cdots + \Delta_0 A^t. \tag{7.1.14}$$

The relation (7.1.14) is exactly (7.1.11). Rewriting (7.1.14) in a matrix form one can derive (7.1.12). In this case, it is shown that (7.1.7) implies (7.1.11) and (7.1.12).

In addition, it follows from the relation (7.1.12) that

$$AL_{t-1} = ACH_{t-1} + A^2 CH_{t-2} + \cdots + A^t CH_0 + \Delta_{t-1} A + \Delta_{t-2} A^2 \cdots + \Delta_0 A^t.$$

Combining this relation with (7.1.11) we can obtain the last expression of (7.1.7). For $i \in I[1, t-1]$, according to (7.1.12) one has

$$\begin{aligned} & AL_{i-1} + CH_i + \Delta_i I_n \\ & = A\Big(\sum_{j=0}^{i-1} [A^{i-j-1} CH_j + \Delta_j A^{i-j-1}]\Big) + CH_i + \Delta_i I_n \\ & = \sum_{j=0}^{i-1} [A^{i-j} CH_j + \Delta_j A^{i-j}] + CH_i + \Delta_i I_n \\ & = \sum_{j=0}^{i} [A^{(i+1)-j-1} CH_j + \Delta_j A^{(i+1)-j-1}] \\ & = L_i. \end{aligned}$$

This is just the second expression of (7.1.7). The first expression of (7.1.7) is obvious. Thus the relation (7.1.11) and (7.1.12) imply the relation (7.1.7). So the proof is completed. □

In the following, the presented closed-form solution to the matrix equation (7.0.1) is in an extremely neat form represented by the controllability matrix associated with the matrices A and C, the observability matrix associated with B and the free parameter matrix Z, and the symmetric operator matrix. This property may propose convenience and advantages to the further analysis of the matrix equation (7.0.1).

Theorem 7.2. Given matrices $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{p\times p}$, $C \in \mathfrak{R}^{n\times r}$ and $R \in \mathfrak{R}^{n\times p}$, suppose that $\{s|\det(I_n - sA) = 0\} \bigcap \lambda(B) = \phi$, and the polynomial matrix

$H(s) = \sum_{i=1}^{t} H_i s^i \in \Re^{r \times n}[s]$ and polynomial $\Delta(s) = \sum_{i=1}^{t} \Delta_i s^i \in R[s]$ satisfy (7.1.11) and $\Delta(s) \neq 0$, for any $s \in \lambda(B)$. Then all the solutions to the matrix equation (7.0.1) can be stated as

$$\begin{cases} X = Ctr_n(A, C) S_n(f_{(I_n, A)}(s) I_r) Obs_n(B, Z) \\ \qquad + Ctr_t(A, C) S_t(H(s) I_r) Obs_t\Big(B, R[\Delta(B)]^{-1}\Big) \\ \qquad + Ctr_t(A, I_n) S_t(\Delta(s) I_n) Obs_t\Big(B, R[\Delta(B)]^{-1}\Big), \\ Y = Z f_{(I_n, A)}(B) + \sum_{i=0}^{t} H_i R[\Delta(B)]^{-1} B^i, \end{cases} \tag{7.1.15}$$

where Z is an arbitrary chosen real parameter matrix.

Proof. According to the relation (7.1.10), we can derive

$$\begin{cases} R_0 = I_n, \\ R_1 = \alpha_1 I_n + A, \\ R_2 = \alpha_2 I_n + \alpha_1 A + A^2, \\ \cdots \\ R_{n-1} = \alpha_{n-1} I_n + \alpha_{n-2} A + \cdots + A^{n-1}. \end{cases}$$

This relation can be compactly expressed as

$$R_i = \sum_{k=0}^{i} \alpha_k A^{i-k}, \alpha_0 = 1, i = 1, 2, \ldots, n-1.$$

Therefore, one can obtain

$$\sum_{i=0}^{n-1} R_i CZB^i = \sum_{i=0}^{n-1} \sum_{k=0}^{i} \alpha_k A^{i-k} CZB^i.$$

With these relations, one easily has

$$\sum_{i=0}^{n-1} R_i CZB^i = Ctr(A, C) S_n(f_{(I_n, A)}(s) I_r) Obs_n(B, Z), \tag{7.1.16}$$

It follows from lemma 7.2 that the relation (7.1.1) holds if and only if the relations (7.1.11) and (7.1.12) hold. So we have

$$\begin{aligned} &\sum_{i=0}^{t-1} L_i R[\Delta(B)]^{-1} B^i \\ &\quad = [L_0 \ L_1 \cdots \ L_{t-1}] Obs_t(B, R[\Delta(B)]^{-1}) \\ &\quad = [Ctr_t(A, C) S_t(H(s)) + Ctr_t(A, I_n) S_t(\Delta(s)) I_n] \\ &\qquad \times Obs_t(B, R[\Delta(B)]^{-1}). \end{aligned} \tag{7.1.17}$$

Substituting (7.1.16) and (7.1.17) into the expression (7.1.3), gives (7.1.15). The proof is completed. □

Example 7.1. In order to show the efficiency of theorem 7.2, we give an example in the form of (7.0.2). We use the following parameter matrices.

$$A = \begin{pmatrix} 1 & 4 & -3 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 4 & 2 & 4 \\ 9 & 6 & 2 \\ 4 & 2 & 1 \end{pmatrix}, C = \begin{pmatrix} 9 & 7 & 2 \\ 4 & 11 & 10 \\ 7 & 1 & -1 \end{pmatrix},$$

we can check $\{s|\det(I - sA) = 0\} \bigcap \lambda(B) = \phi$.

By simple computations, it is easy to obtain

$$f_{(I_3,A)}(s) = \det(I - sA) = 6s^3 - 4s^2 - 3s + 1,$$

$$S_3(f_{(I_3,A)(s)}, I_3) = \begin{pmatrix} 1 & 0 & 0 & -3 & 0 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 & -3 & 0 & 0 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$Ctr_3(A, C) = \begin{pmatrix} 9 & 7 & 2 & 4 & 48 & 45 & 62 & 138 & 102 \\ 4 & 11 & 10 & 13 & 18 & 12 & 17 & 66 & 57 \\ 7 & 1 & -1 & -2 & -6 & -3 & -6 & -54 & -48 \end{pmatrix},$$

$$Obs_3(B, Z) = \begin{pmatrix} -1.2808 & 1.8309 & -3.0644 \\ 4.0899 & -5.9867 & 10.1498 \\ -3.8419 & 5.7190 & -9.7834 \\ -0.9027 & 2.2950 & -4.5257 \\ 3.0786 & -7.4407 & 14.5360 \\ -3.0300 & 7.0635 & -13.7130 \\ -1.0587 & 2.9131 & -3.5466 \\ 3.4920 & -9.4151 & 11.9690 \\ -3.4004 & 8.8951 & -11.7060 \end{pmatrix}.$$

Choose $Z = \begin{pmatrix} -1.2808 & 1.8309 & -3.0644 \\ 4.0899 & -5.9867 & 10.1498 \\ -3.8419 & 5.7190 & -9.7834 \end{pmatrix}$, so from theorem 7.2 we have

$$X = \begin{pmatrix} 1 & 4 & 9 \\ 2 & 7 & 0 \\ 1 & -1 & 2 \end{pmatrix}, Y = \begin{pmatrix} 52.4426 & 32.9016 & 12.9672 \\ -156.4645 & -99.4153 & -38.6940 \\ 136.6339 & 87.8962 & 33.0765 \end{pmatrix}.$$

When the matrix triple (A, I_n, C) is $R-$controllable, that is $\mathrm{rank}[I_n - sA \ \ -C] = n$, for any $s \in R$, the approach of theorem 7.2 can be readily employed to solve the matrix equation (7.0.1) without use of Smith normal form reduction. This is summarized as the following corollary.

Corollary 7.1. Given matrices $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{p\times p}$, $C \in \mathfrak{R}^{n\times r}$ and $R \in \mathfrak{R}^{n\times p}$, suppose that $\{s|\det(I_n - sA) = 0\} \bigcap \lambda(B) = \phi$ and (A, I_n, C) is $R-$controllable. Let the polynomial matrix $H(s) = \sum_{i=0}^{t} H_i s^i \in \mathfrak{R}^{r\times n}[s]$ satisfy

$$\sum_{i=0}^{t} A^{t-i} C H_i = -A^t. \tag{7.1.18}$$

Then all the solutions to the matrix equation (7.0.1) can be characterized by

$$\begin{cases} X = Ctr_t(A, C) S_n(f_{(I_n,A)}(s) I_r) Obs_n(B, Z) \\ \qquad + [Ctr_t(A, C) S_t(H(s)) + Ctr_t(A, I_n)] Obs_t(B, R), \\ Y = Z f_{(I_n,A)}(B) + \sum\limits_{i=0}^{t} H_i R B^i, \end{cases} \tag{7.1.19}$$

in which Z is an arbitrary chosen parameter matrix.

According to theorem 7.2, when (A, I_n, C) is $R-$controllable, this corollary can be derived if the polynomial $\Delta(s)$ choose to be 1. In this case, the relations $[\Delta(B)]^{-1} = I_n$ and $S_t(\Delta(s) I_n) = I$ hold, and the matrix equation (7.1.11) transforms into the the matrix equation (7.1.18). If we denote $H_{[t]} = [H_t^T \ H_{t-1}^T \ \cdots \ H_0^T]^T$, this equation can be rewritten as

$$Ctr_{t+1}(A, C) H_{[t]} = -A^t. \tag{7.1.20}$$

If the number t is sufficiently large, one gives

$$\mathrm{rank}[Ctr_{t+1}(A, C)] = \mathrm{rank}[Ctr_{t+1}(A, C) \ \ A^t]. \tag{7.1.21}$$

Under this condition, the matrices H_i, $i \in [1, t]$, can be obtained from (7.1.19) by using some numerically reliable approaches such as QR decomposition and SVD decomposition. Furthermore, the relation $S_t(\Delta(s) I_n) = 0$ holds when the polynomial $\Delta(s)$ is selected to be s^t. Similar to the above argument, it is easy to derive the following corollary.

Corollary 7.2. Let $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{p\times p}$ and $C \in \mathfrak{R}^{n\times r}$, suppose that $\{s|\det(I_n - sA) = 0\} \bigcap \lambda(B) = \phi$ and 0 is not included the set $\lambda(B)$. If the polynomial matrix $H(s) = \sum_{i=0}^{t} H_i s^i \in \mathfrak{R}^{r\times n}[s]$ satisfy

$$\sum_{i=0}^{t} A^{t-i} C H_i = -I_n. \tag{7.1.22}$$

Then all the solutions to the matrix equation (7.0.1) can be stated as

$$\begin{cases} X = Ctr_t(A, C)S_n(f_{(I_n,A)}(s)I_r)Obs_n(B, Z) \\ \qquad + Ctr_t(A, C)S_t(H(s))Obs_n(B, RB^{-t}), \\ Y = Zf_{(I_n,A)}(B) + \sum\limits_{i=0}^{t} H_iRB^i, \end{cases} \tag{7.1.23}$$

where Z is an arbitrary chosen parameter matrix.

7.2 Parametric Pole Assignment for Descriptor Linear Systems by P-D Feedback

Example 7.2. As an application of the proposed solution, we consider the following linear system:

$$E\dot{x} = Ax + Bu, \tag{7.2.1}$$

where $A, E \in R^{n\times n}$ and $B \in R^{n\times r}$ are known coefficient matrices. If the following state feedback controller

$$u = -K_px - K_d\dot{x}, \tag{7.2.2}$$

is applied to the above system (7.2.1), the closed-loop system becomes

$$(E + CK_d)\dot{x} = (A - CK_p)x. \tag{7.2.3}$$

Thus, the closed-loop system (7.2.3) is regular if and only if

$$\det(E + BK_d) \neq 0. \tag{7.2.4}$$

In this case, if the closed-loop system is supposed to be regular, the closed-loop system (7.2.3) can be rewritten as

$$\dot{x} = A_cx, \quad A_c = (E + BK_d)^{-1}(A - BK_p), \tag{7.2.5}$$

Given a set of desired closed-loop poles $s_i, i = 1, 2, \ldots, n$. The parametric pole assignment problem is to determine the matrices K_d and K_p such that the closed-loop system matrix $(E + BK_d)^{-1}(A - BK_p)$ has desired eigenvalues and eigenstructure, *i.e.* determine the matrices K_d and K_p such that

$$K_p = (Y - K_dXB)X^{-1}, \tag{7.2.6}$$

and eigenstructure

$$\sigma(A_c) = \sigma((E + BK_d)^{-1}(A - BK_p)) = \{s_i, i = 1, 2, \ldots, n\}, \tag{7.2.7}$$

Note that (7.2.7) is satisfied if and only if there exists a nonsingular matrix X and a matrix B satisfying $\sigma(A_c) = \sigma(B)$ such that

$$X^{-1}[(E + BK_d)^{-1}(A - BK_p)]X = F,$$

which is equivalent to the generalized Sylvester matrix equation (7.0.3) by setting $Y = K_pX + K_dXF$. Therefore, the pivotal step in solving the parametric pole assignment problem is to solve the generalized Sylvester matrix equation (7.0.3). The following lemma shows that the generalized Sylvester matrix equation (7.0.3) is the equivalent form of the generalized discrete Sylvester matrix equation (7.0.2).

Lemma 7.3. [81]The generalized Sylvester matrix equation (7.0.3), where $A, E \in \Re^{n\times n}$, $B \in \Re^{n\times r}$, and $F \in \Re^{p\times p}$ are known and the matrix pair (E, A) is regular, is equivalent to the generalized discrete Sylvester matrix equation (7.0.2) with

$$M = (\gamma E - A)^{-1}E, \quad N = \gamma I - F, \quad T = (\gamma E - A)^{-1}B,$$

where γ is an arbitrary scalar such that $(\gamma E - A)$ is nonsingular.

Lemma 7.4. [81]There exists a matrix K_d such that $\det(E + BK_d) \neq 0$ holds if and only if $\operatorname{rank}[E \quad B] = n$ holds. Further, let $\operatorname{rank}[E \quad B] = n$ hold and $U \in R^{n\times n}$ be a nonsingular matrix such that

$$UB = \begin{pmatrix} \Pi_b \\ 0 \end{pmatrix}, \quad UE = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix},$$

where $\Pi_b \in \Re_r^{r\times r}$, then all the matrix K_d satisfying $\det(E + BK_d) \neq 0$ can be characterized as

$$K_d = \Pi_b^{-1}(P - E_1),$$

with $P \in R_r^{r\times n}$ satisfying

$$Im(E_2^T t) \oplus Im(P^T) = R^n.$$

Generally speaking, if the desired eigenvalue set contains complex numbers, the coefficient matrix B is a complex matrix. Thus the existing results involve complex arithmetic, for example in[127]. However, complex arithmetic can be avoided if the results of this chapter are used. In order to illustrate our approach to solving the generalized discrete Sylvester matrix equation $X - AXB = CY$ with their applications on parametric pole assignment for descriptor linear systems. In the following, we consider a system in the form of (7.0.3) with the following system matrices (borrowed from[129,135]).

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is obvious that such a system is C-controllable. If we choose $\gamma = 1$, then according to lemma 7.3, we can get the matrices M and N as

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ -1 & -1 \\ 0 & 0 \end{pmatrix}.$$

The inverse characteristic polynomial of M is

$$\beta(s, \psi) = \beta(s, 3) = \det(sM - I) = s^3 + 1.$$

Therefore, we have

$$d(s, 6) = 0s^6 + 0s^5 + 0s^4 + s^3 + 1$$

and

$$S_2(I_2, A) = \begin{pmatrix} I_2 & 0I_2 & 0I_2 & I_2 & 0I_2 & 0I_2 \\ & I_2 & 0I_2 & 0I_2 & I_2 & 0I_2 \\ & & I_2 & 0I_2 & 0I_2 & I_2 \\ & & & I_2 & 0I_2 & 0I_2 \\ & & & & I_2 & 0I_2 \\ & & & & & I_2 \end{pmatrix}.$$

We now consider a P-D feedback law in the form of (7.2.1). For select the closed-loop eigenvalue set as

$$\sigma(A_c) = \sigma(F) = \{-1, -1, -2, -2, -1 \pm i\}.$$

Thus, the matrix F can be chosen as

$$F = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & -2 \end{pmatrix}, \quad N = \gamma I_6 - F = -F.$$

Under this condition, according to theorem 7.2, a complete parametric solution to the generalized discrete Sylvester matrix equation (7.0.2) can be expressed as

$$\begin{cases} X = Ctr_6(M, T) S_2(I_2, A) Obs_6(Z, N), \\ Y = -Z - ZN^3. \end{cases}$$

where $Z \in R^{2\times 6}$ is an arbitrary parametric matrix. In order to guarantee that X is nonsingular, we need to have

$$(z_{25}^2 - 2z_{26}z_{25} + 2z_{26}^2)(z_{22}z_{11} - z_{21}z_{12})(-z_{14}z_{23} + z_{13}z_{24}) \neq 0.$$

In addition, the matrices U, E_1, E_2 and $\mathbf{\Pi}_b$ defined in lemma 7.4 can be expressed as

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{\Pi}_b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Therefore, by lemma 7.4 all the derivative gain matrix K_d can be parameterized as

$$\begin{aligned} K_d &= \mathbf{\Pi}_b^{-1}(P - E_1) \\ &= \begin{pmatrix} p_{11} - 1 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} + 1 - p_{11} & p_{22} - p_{12} & p_{23} {-} 13 & p_{24} - p_{14} & p_{25} - p_{15} & p_{26} - p_{16} \end{pmatrix}, \end{aligned}$$

where $P \in R^{2\times 6}$ is a parameter matrix satisfying

$$p_{26}p_{14} - p_{16}p_{24} \neq 0.$$

According to (7.2.5), by specially choosing

$$Z = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can obtain

$$K_d = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \quad K_p = \begin{pmatrix} -11 & -13 & -5 & 3 & 8 & -2 \\ 27 & 23 & 9 & -3 & -11 & 8 \end{pmatrix}.$$

Example 7.3. Consider the following linear system:

$$E\dot{x} = Ax + Bu, \tag{7.2.8}$$

where $A, E \in R^{n\times n}$ and $B \in R^{n\times r}$ are known coefficient matrices. If the following state feedback controller

$$u = -K_p x - K_d \dot{x}, \tag{7.2.9}$$

is applied to the above system (7.2.8), the closed-loop system becomes

$$(E + BK_d)\dot{x} = (A - BK_p)x, \tag{7.2.10}$$

The eigenstructure assignment problem is to determine the matrix K_p such that the closed-loop system matrix $A - BK_p$ has desired eigenvalues and struture, *i.e.*, determine the matrix K_p such that

$$A - BK_p = EXFX^{-1}, \tag{7.2.11}$$

where F is the desired Jordan form of the closed-loop system and X is the corresponding eigenvector matrix. If we choose $K_pX + K_dXF = Y$, the equation (7.2.11) is equivalent to the generalized Sylvester matrix equation $AX - EXF = BY$. By lemma 7.3, the generalized Sylvester matrix equation $AX - EXF = BY$ is equivelent into the Yakubovich matrix equation $X - MXN = TY$. By theorem 7.1, we can obtain the solution to Yakubovich matrix equation $X - MXN = TY$.

If we choose $A = \begin{pmatrix} -2 & -4 & -3 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} 4 & 3 & 2 \\ 9 & 8 & 7 \\ 4 & 3 & 2 \end{pmatrix}$, $F = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 9 & 7 & 6 \\ 4 & 3 & 2 \\ 0 & 1 & 0 \end{pmatrix}$, we can check $\{s|\det(I - sA) = 0\} \bigcap \lambda(B) = \phi$. If we choose $\gamma = 1$, by lemma 7.3 we have

$$M = \begin{pmatrix} 1.4400 & 1.2800 & 1.1200 \\ -1.4200 & -1.5400 & -1.6600 \\ 1.0600 & 1.2200 & 1.3800 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & -2 & -3 \\ 0 & 0 & -2 \\ -4 & 0 & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} -1.8800 & -1.2000 & -1.3600 \\ 3.3400 & 2.6000 & 2.4800 \\ -0.6200 & -0.8000 & -0.6400 \end{pmatrix}.$$

By simple computations, it is easy to obtain $f_{(I_3,A)}(s) = \det(I - sM) = 0.3s^2 - 1.28s + 1$,

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0.16 & 1.28 & 1.12 \\ -1.42 & -2.82 & -1.66 \\ 1.06 & 1.22 & 0.1 \end{pmatrix},$$
$$R_2 = \begin{pmatrix} -0.1 & -0.4 & -0.4 \\ 0.2 & 0.8 & 0.8 \\ -0.1 & -0.4 & -0.4 \end{pmatrix}.$$

Choose $Z = \begin{pmatrix} -106.91 & -38.1915 & -106.3752 \\ -70.0416 & -29.1167 & -71.1143 \\ 230.1276 & 82.4103 & 228.1911 \end{pmatrix}$, so from theorem 7.1 we have

$$X = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 6 & 3 \\ 4 & 2 & 1 \end{pmatrix}, Y = \begin{pmatrix} -189.3333 & -110.6667 & -243.3333 \\ -130.0000 & -76.0000 & -159.0000 \\ 410.1667 & 236.3333 & 520.6667 \end{pmatrix}.$$

7.3 Conclusions

In this chapter, the solutions to the nonhomogeneous Yakubovich matrix equation in the real field are presented by means of Smith's normal form reduction and polynomial matrix. These proposed solutions can provide all the degrees of freedom, which are represented by an arbitrarily chosen parameter matrix Z. All the coefficient matrices are not restricted to be in any canonical form. The matrix B explicitly appears in the solutions, thus can be unknown *a priori*. From the discussion in this chapter, one can observe that the solutions to the nonhomogeneous Yakubovich matrix equation are crucial as the theoretical basis of the development of many kinds of other matrix equations, and are deserved further investigation in the future. In addition, as the theoretical generalization of the well-known generalized Sylvester matrix equation, it may be helpful for future control applications.

Chapter 8

Explicit Solutions to the Nonhomogeneous Yakubovich-Transpose Matrix Equation

The well-known Karm-Yakubovich-transpose matrix equation

$$X - AX^T B = C, \tag{8.0.1}$$

and the general discrete Lyapunov-transpose matrix equation

$$X - B^T X^T B = CC^T, \tag{8.0.2}$$

are closely related to many problems in conventional linear control system theory, such as pole/eigenstructure assignment design, Luenberger-type observer design, robust fault detection, and so on. Moreover, the generalized discrete Yakubovich-transpose matrix equation

$$X - AX^T B = CY, \tag{8.0.3}$$

where $A \in \Re^{n\times p}$, $B \in \Re^{n\times p}$ and $C \in \Re^{n\times r}$ are the known matrices and $X \in \Re^{n\times p}$, $Y \in R^{r\times p}$ are the unknown matrices, has important applications in dealing with complicated linear systems, such as large-scale systems with interconnections, linear systems with certain partitioned structures or extended models and second order or higher order linear systems. As a generalization of the above transpose matrix equations, the following nonhomogeneous Yakubovich-transpose matrix equation

$$X - AX^T B = CY + R, \tag{8.0.4}$$

is considered in this chapter, where $A \in \Re^{n\times p}$, $B, R \in \Re^{n\times p}$ and $C \in \Re^{n\times r}$ are the given matrices, and $X \in \Re^{n\times p}$, $Y \in \Re^{r\times p}$ are the unknown matrices. In the present chapter, two methods to obtain the solution to the nonhomogeneous Yakubovich-transpose matrix equation are given. Furthermore, the equivalent form

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of the solution is proposed. One of the solutions is established with the controllability matrix, the observability matrix, and symmetric operator matrix.

8.1 The First Approach

To obtain the result, we need the following lemma with the aid of Smith normal form reduction.

Lemma 8.1. Given matrices $A \in \mathfrak{R}^{n\times p}$, $B \in \mathfrak{R}^{n\times p}$ and $C \in \mathfrak{R}^{n\times r}$, suppose that $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda\{B^TA\} = \phi$. Then there exist polynomial matrices $L(s) \in \mathfrak{R}^{n\times n}[s]$, $H(s) \in \mathfrak{R}^{r\times n}[s]$, and polynomial $\Delta(s) \in R[s]$ such that

$$(I - sAB^T)L(s) - CH(s) = \Delta(s)I_n, \tag{8.1.1}$$

where $\Delta(\gamma) \neq 0$, for any $\gamma \in \lambda(B^TA)$.

Proof. Let $P(s) \in \mathfrak{R}^{n\times n}[s]$ and $Q(s) \in \mathfrak{R}^{(n+r)\times(n+r)}[s]$ be two uniocular polynomial matrices that transform $[I_n - sAB^T \ - C]$ into its Smith normal form, that is,

$$P(s)[I_n - sAB^T \ - C]Q(s) = [\operatorname{diag}_{i=1}^n d_i(s) \ 0\ ], \tag{8.1.2}$$

with $d_{i-1}(s)|d_i(s) \in R[s], i \in I[2,n]$. Since $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda\{B^TA\} = \phi$, then $d_i(\gamma) \neq 0, i \in I[1,n]$, for any $\gamma \in \lambda(B^TA)$. Partition $Q(s)$ into $[Q_1(s), Q_2(s)]$ with $Q_1(s) \in \mathfrak{R}^{(n+r)\times n}[s]$. It follows from (8.1.2) that

$$\begin{aligned}
&P(s)[I_n - sAB^T \ - C]Q_1(s) = \operatorname{diag}_{i=1}^n d_i(s)\\
&\iff P(s)[I_n - sAB^T \ - C]Q_1(s)\operatorname{diag}_{i=1}^n \frac{d_n(s)}{d_i(s)} = I_n d_n(s)\\
&\iff [I_n - sAB^T \ - C](Q_1(s)\operatorname{diag}_{i=1}^n \frac{d_n(s)}{d_i(s)})P(s) = I_n d_n(s).
\end{aligned}$$

Thus, one can choose $\Delta(s) = d_n(s)$ and

$$\begin{bmatrix} L(s) \\ H(s) \end{bmatrix} = \left(Q_1(s)\operatorname{diag}_{i=1}^n \frac{d_n(s)}{d_i(s)}\right)P(s).$$

to satisfy the relation (8.1.1). □

With this lemma, now we are in a position to provide a closed-form solution of the nonhomogeneous Yakubovich-transpose matrix equation (8.0.4).

Theorem 8.1. Let $A \in \mathfrak{R}^{n\times p}$, $B \in \mathfrak{R}^{n\times p}$ and $C \in \mathfrak{R}^{n\times r}$, suppose that $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda(B^TA) = \phi$, and polynomial matrices $L(s) = \sum_{i=0}^{t-1} L_i s^i \in \mathfrak{R}^{n\times n}[s]$ and $H(s) = \sum_{i=0}^{t} H_i s^i \in \mathfrak{R}^{r\times n}[s]$, and polynomial $\Delta(s) = \sum_{i=0}^{t} \Delta_i s^i \in R[s]$ satisfy (8.1.1) with $\Delta(\gamma) \neq 0$, for any $\gamma \in \lambda(B^TA)$. Further, let

$$f_{(I,AB^T)}(s) = \det(I - sAB^T) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0, \alpha_0 = 1,$$

and

$$\mathrm{adj}(I - sAB^T) = R_{n-1}s^{n-1} + \cdots + R_1 s + R_0.$$

Then all the solutions to the nonhomogeneous Yakubovich-transpose matrix equation (8.0.1) can be characterized by

$$\begin{cases} X = \sum_{i=0}^{n-1} R_i CZ(A^TB)^i + \sum_{i=0}^{n-1} A(B^TA)^i(R_iCZ)^TB + \sum_{i=0}^{t-1} L_i \\ \qquad \times R[\Delta(A^TB)]^{-1}(A^TB)^i + \sum_{i=0}^{t-1} A(B^TA)^i(L_iR[\Delta(A^TB)]^{-1})^TB, \\ Y = Zf_{(I,AB^T)}(A^TB) + \sum_{i=0}^{t} H_iR[\Delta(A^TB)]^{-1}(A^TB)^i. \end{cases} \tag{8.1.3}$$

Proof. We first show that the matrices X and Y given in (8.1.3) are solutions of the matrix equation (8.0.4). A direct calculation gives

$$\begin{aligned} & X - AX^TB \\ &= \sum_{i=0}^{n-1} R_iCZ(A^TB)^i + \sum_{i=0}^{n-1} A(B^TA)^i(R_iCZ)^TB + \sum_{i=0}^{t-1} L_iR[\Delta(A^TB)]^{-1}(A^TB)^i \\ &\quad + \sum_{i=0}^{t-1} A(B^TA)^i(L_iR[\Delta(A^TB)]^{-1})^TB - \sum_{i=0}^{n-1} A(B^TA)^i(R_iCZ)^TB \\ &\quad - \sum_{i=0}^{n-1} AB^T(R_iCZ)(A^TB)^iA^TB - \sum_{i=0}^{t-1} A(B^TA)^i(L_iR[\Delta(A^TB)]^{-1})^TB \\ &\quad - \sum_{i=0}^{t-1} AB^T(L_iR[\Delta(A^TB)]^{-1})(A^TB)^iA^TB \\ &= \sum_{i=0}^{n-1} R_iCZ(A^TB)^i - \sum_{i=0}^{n-1}(AB^T)R_iCZ(A^TB)^{i+1} + \sum_{i=0}^{t-1} L_i \\ &\quad \times R[\Delta(A^TB)]^{-1}(A^TB)^i - \sum_{i=0}^{t-1} AB^TL_iR[\Delta(A^TB)]^{-1}(A^TB)^{i+1} \\ &= R_0CZ + \sum_{i=1}^{n-1}(R_i - AB^TR_{i-1})CZ(A^TB)^i - AB^TR_{n-1}CZ(A^TB)^n + L_0R[\Delta(A^TB)]^{-1} \\ &\quad + \sum_{i=1}^{t-1}(L_i - AB^TL_{i-1})R[\Delta(A^TB)]^{-1}(A^TB)^i - AB^TL_{t-1}R[\Delta(A^TB)]^{-1}(A^TB)^t. \end{aligned} \tag{8.1.4}$$

By the relation $(I - sAB^T)\mathrm{adj}(I - sAB^T) = I\det(I - sAB^T)$, it is easily derived that

$$\begin{cases} R_0 = \alpha_0 I, \\ R_i - AB^T R_{i-1} = \alpha_i I, \quad i \in I[1, n-1], \\ -AB^T R_{n-1} = \alpha_n I. \end{cases} \tag{8.1.5}$$

So we have

$$\begin{aligned} & R_0 CZ + \sum_{i=1}^{n-1} (R_i - AB^T R_{i-1}) CZ (A^T B)^i - AB^T R_{n-1} CZ (A^T B)^n \\ & \quad = CZ \sum_{i=0}^{n} \alpha_i (A^T B)^i = CZ f_{(I, AB^T)}(A^T B). \end{aligned} \tag{8.1.6}$$

In addition, comparing the power of s in (8.1.1) yields

$$\begin{cases} L_0 = CH_0 + \Delta_0 I, \\ L_i - AB^T L_{i-1} = CH_i + \Delta_i I, \quad i \in I[1, t-1], \\ -AB^T L_{t-1} = CH_t + \Delta_t I. \end{cases} \tag{8.1.7}$$

With these relations, we can easily obtain

$$\begin{aligned} & L_0 R[\Delta(A^T B)]^{-1} + \sum_{i=1}^{t-1} (L_i - AB^T L_{i-1}) R[\Delta(A^T B)]^{-1} (A^T B)^i \\ & \quad - AB^T L_{t-1} R[\Delta(A^T B)]^{-1} (A^T B)^t \\ & \qquad = (CH_0 + \Delta_0 I) R[\Delta(A^T B)]^{-1} + \sum_{i=1}^{t-1} (CH_i + \Delta_i I) \\ & \qquad\quad \times R[\Delta(A^T B)]^{-1} (A^T B)^i + (CH_t + \Delta_t I) R[\Delta(A^T B)]^{-1} (A^T B)^t \\ & \qquad = \sum_{i=0}^{t} CH_i R[\Delta(A^T B)]^{-1} (A^T B)^i + \sum_{i=0}^{t} \Delta_i R[\Delta(A^T B)]^{-1} (A^T B)^i \\ & \qquad = \sum_{i=0}^{t} CH_i R[\Delta(A^T B)]^{-1} (A^T B)^i + R[\Delta(A^T B)]^{-1} \sum_{i=0}^{t} \Delta_i (A^T B)^i \\ & \qquad = \sum_{i=0}^{t} CH_i R[\Delta(A^T B)]^{-1} (A^T B)^i + R[\Delta(A^T B)]^{-1} \Delta(A^T B) \\ & \qquad = \sum_{i=0}^{t} CH_i R[\Delta(A^T B)]^{-1} (A^T B)^i + R. \end{aligned} \tag{8.1.8}$$

Substituting (8.1.6) and (8.1.8) into (8.1.4), yields

$$\begin{aligned}
&X - AX^TB \\
&\quad = CZf_{(I,AB^T)}(A^TB) + \sum_{i=0}^{t} CH_iR[\Delta(A^TB)]^{-1}(A^TB)^i + R \\
&\quad = C\left(Zf_{(I,AB^T)}(A^TB) + \sum_{i=0}^{t} H_iR[\Delta(A^TB)]^{-1}(A^TB)^i\right) + R \\
&\quad = CY + R.
\end{aligned}$$

Therefore, the matrices X and Y given in (8.1.3) satisfy the matrix equation (8.0.4).

Next, the completeness of solution (8.1.3) is showed. First of all, from the relation (8.1.3) we can see the solution (8.1.3) has rp parameters represented by the elements of the free matrix Z. Because $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda(B^TA) = \phi$, $f_{(I,AB^T)}(A^TB)$ is nonsingular. So it is showed that the mapping $Z \to (X, Y)$ defined by (8.1.3) is injective. The proof is thus completed. □

Regarding the solution of $\alpha_i, i \in I[0, n]$, and $R_i, i \in I[0, n-1]$, the so-called generalized Leverrier algorithm[139] can be stated as the following relation:

$$\begin{cases} R_i = AB^TR_{i-1} + \alpha_iI, \quad i \in I[1, n-1], \quad R_0 = I, \\ \alpha_i = \dfrac{\operatorname{trace}(AB^TR_{i-1})}{i}. \end{cases} \quad i \in I[1, n]. \tag{8.1.9}$$

For relative lower-order cases, the Smith norm form reduction can be realized by using elementary polynomial matrix transformation. For high-order cases, the Maple function “smith” can be readily employed.

From theorem 8.1, it is easy to know that under the condition of $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda(B^TA) = \phi$ all the solutions of the matrix equation (8.0.3) can be explicitly given by

$$\begin{cases} X = \sum\limits_{i=0}^{n-1} R_iCZ(A^TB)^i + \sum\limits_{i=0}^{n-1} A(B^TA)^i(R_iCZ)^TB, \\ Y = Zf_{(I,AB^T)}(A^TB). \end{cases} \tag{8.1.10}$$

Moreover, from theorem 8.1 it is obvious that a special solution of the matrix equation (8.0.4) is

$$\begin{cases} X = \sum\limits_{i=0}^{t-1} L_iR[\Delta(A^TB)]^{-1}(A^TB)^i \\ \qquad + \sum\limits_{i=0}^{t-1} A(B^TA)^i(L_iR[\Delta(A^TB)]^{-1})^TB, \\ Y = \sum\limits_{i=0}^{t} H_iR[\Delta(A^TB)]^{-1}(A^TB)^i. \end{cases}$$

Similarly to the reference[145], about theorem 8.1 we have the following equivalent form. First we give the following lemma. The idea of the proof of this lemma is motivated by[148].

Lemma 8.2. Suppose that $L_i, i \in I[0, t-1]$, and $H_i, \Delta_i, i \in I[0, t]$, be the coefficients of $L(s)$, $H(s)$ and $\Delta(s)$, respectively. Then they satisfy (8.1.1) if and only if $H_i, \Delta_i, i \in I[0, t]$, satisfy

$$\sum_{i=0}^{t} (AB^T)^{t-i} CH_i = -\sum_{i=0}^{t} \Delta_i (AB^T)^{t-i}, \tag{8.1.11}$$

and $L_i, i \in I[0, t-1]$, satisfy

$$[L_0 \ L_1 \ \cdots \ L_{t-1}] = Ctr_t(AB^T, C)S_t(H(s)) + Ctr_t(AB^T, I_n)S_t(\Delta(s)I_n). \tag{8.1.12}$$

Proof. From the proof of theorem 8.1 we can see that $L_i, i \in I[0, t-1]$, and $H_i, \Delta_i, i \in I[0, t]$ satisfy (8.1.1) if and only if they satisfy (8.1.7). So we need to show that the relation (8.1.7) is equivalent to the relations (8.1.11) and (8.1.12).

From the recursive relation (8.1.7), a direct calculation gives

$$\begin{cases} L_0 = CH_0 + \Delta_0 I, \\ L_1 = CH_1 + AB^T CH_0 + \Delta_1 I + AB^T \Delta_0 I, \\ \cdots \\ L_{t-1} = CH_{t-1} + AB^T CH_{t-2} + \cdots + (AB^T)^{t-1} CH_0 \\ \qquad\quad + \Delta_{t-1} I + \Delta_{t-2} AB^T + \cdots + \Delta_0 (AB^T)^{t-1}. \end{cases} \tag{8.1.13}$$

and

$$0 = CH_t + AB^T CH_{t-1} + \cdots + (AB^T)^t CH_0 + \Delta_t I + \Delta_{t-1} AB^T + \cdots + \Delta_0 (AB^T)^t. \tag{8.1.14}$$

The relation (8.1.14) is exactly (8.1.11). Rewriting (8.1.13) in a form gives (8.1.12). So it is shown that (8.1.7) implies (8.1.11) and (8.1.12).

On the other hand, it follows from the relation (8.1.12) that

$$\begin{aligned} &AB^T L_{t-1} \\ &\quad = AB^T CH_{t-1} + (AB^T)^2 CH_{t-2} + \cdots + (AB^T)^t CH_0 \\ &\qquad + \Delta_{t-1} AB^T + \cdots + \Delta_0 (AB^T)^t. \end{aligned}$$

Combining this relation with (8.1.11) we can obtain the last expression of (8.1.7). For $i \in I[1, t-1]$, from (8.1.12) one has

$$\begin{aligned} &AB^T L_{i-1} + CH_i + \Delta_i I \\ &\quad = AB^T \Big(\sum_{j=0}^{i-1} [(AB^T)^{i-j-1} CH_j + \Delta_j (AB^T)^{i-j-1}] \Big) + CH_i + \Delta_i I \\ &\quad = \sum_{j=0}^{i-1} [(AB^T)^{i-j} CH_j + \Delta_j (AB^T)^{i-j}] + CH_i + \Delta_i I \\ &\quad = \sum_{j=0}^{i} [(AB^T)^{(i+1)-j-1} CH_j + \Delta_j (AB^T)^{(i+1)-j-1}] \\ &\quad = L_i. \end{aligned}$$

This is the second expression of (8.1.7). The first expression of (8.1.7) is obvious. Thus the relation (8.1.11) and (8.1.12) imply the relation (8.1.7). So the proof is completed. □

In the following theorem, the solution to the matrix equation (8.0.4) is stated as the controllability matrix associated with the matrices AB^T and C, and the observability matrix associated with B and the free parameter matrix. This property may propose convenience and advantages to the further analysis of the matrix equation (8.0.4).

Theorem 8.2. Let the matrices $A \in \Re^{n\times p}$, $B \in \Re^{n\times p}$ and $C \in \Re^{n\times r}$ be given, suppose that $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda\{B^TA\} = \phi$, and the polynomial matrix $H(s) = \sum_{i=1}^{t} H_i s^i \in \Re^{r\times n}$ and polynomial $\Delta(s) = \sum_{i=1}^{t}\Delta_i s^i \in R[s]$ satisfy (8.1.10) and $\Delta(\gamma) \neq 0$, for any $\gamma \in \lambda(B^TA)$. Then all the solution to the matrix equation (8.0.1) can be stated as

$$\begin{cases} X = \mathit{Ctr}(AB^T, C)S_n(f_{(I,AB^T)}(s)I_r)\mathit{Obs}_n(A^TB, Z) \\ \qquad + A[\mathit{Ctr}(AB^T, C)S_n(f_{(I,AB^T)}(s)I_r)\mathit{Obs}_n(A^TB, Z)]^TB \\ \qquad + [\mathit{Ctr}(AB^T, C)S_t(H(s)) + \mathit{Ctr}(AB^T, I_n)S_t(\Delta(s)I_n)] \\ \qquad \times\ \mathit{Obs}_t(A^TB, R[\Delta(A^TB)]^{-1}) \\ \qquad + A\{[\mathit{Ctr}(AB^T, C)S_t(H(s)) + \mathit{Ctr}_t(AB^T, I_n)S_t(\Delta(s))I_n] \\ \qquad \mathit{Obs}_t(A^TB, R[\Delta(A^TB)]^{-1})\}^TB, \\ Y = Zf_{(I,AB^T)}(A^TB) + \sum_{i=1}^{t} H_iR[\Delta(A^TB)]^{-1}(A^TB)^i, \end{cases} \tag{8.1.15}$$

where Z is an arbitrary chosen parameter matrix.

Proof. By the relation (8.1.9), it is easily obtained that

$$\begin{cases} R_0 = I_n, \\ R_1 = \alpha_1 I_n + AB^T, \\ R_2 = \alpha_2 I_n + \alpha_1 AB^T + (AB^T)^2, \\ \cdots \\ R_{n-1} = \alpha_{n-1}I_n + \alpha_{n-2}AB^T + \cdots + (AB^T)^{n-1}. \end{cases}$$

This relation can be compactly expressed as

$$R_i = \sum_{k=0}^{i} \alpha_k (AB^T)^{i-k}, \alpha_0 = 1, i = 1, 2, \ldots, n-1.$$

Therefore, it is easily obtained

$$\sum_{i=0}^{n-1} R_iCZ(AB^T)^i = \sum_{i=0}^{n-1}\sum_{k=0}^{i} \alpha_k(AB^T)^{i-k}CZ(AB^T)^i.$$

So we have

$$\sum_{i=0}^{n-1} R_i CZ(A^T B)^i = Ctr(AB^T, C)S_n(f_{(I,AB^T)}(s)I_r)\,Obs_n(A^T B, Z), \tag{8.1.16}$$

and

$$\begin{aligned}\sum_{i=0}^{n-1}(B^T A)^i(R_i CZ)^T B &= \sum_{i=0}^{n-1}[R_i CZ(A^T B)^i]^T B\\ &= [Ctr(AB^T, C)S_n(f_{(I,AB^T)}(s)I_r)\,Obs_n(A^T B, Z)]^T B.\end{aligned} \tag{8.1.17}$$

It follows from lemma 8.2 that the relation (8.1.1) holds if and only if the relations (8.1.11) and (8.1.12) hold. So we have

$$\begin{aligned}&\sum_{i=0}^{t-1} L_i R[\Delta(A^T B)]^{-1}(A^T B)^i\\ &\quad= [L_0\ L_1\ L_{t-1}]\,Obs_t(A^T B, R[\Delta(A^T B)]^{-1})\\ &\quad= [Ctr(AB^T, C)S_t(H(s)) + Ctr(AB^T, I_n)S_t(\Delta(s))I_n]\,Obs_t(A^T B, R[\Delta(A^T B)]^{-1}).\end{aligned} \tag{8.1.18}$$

Similarly, one has

$$\begin{aligned}\sum_{i=0}^{t-1}(B^T A)^i(L_i R[\Delta(A^T B)]^{-1})^T B &= \Big[\sum_{i=0}^{t-1} L_i R[\Delta(A^T B)]^{-1}(A^T B)^i\Big]^T B\\ &= \{[Ctr(AB^T, C)S_t(H(s)) + Ctr_t(AB^T, I_n)S_t(\Delta(s))I_n]\\ &\quad\times Obs_t(A^T B, R[\Delta(A^T B)]^{-1})\}^T B.\end{aligned} \tag{8.1.19}$$

Substituting (8.1.16)–(8.1.19) into the expression (8.1.3), gives (8.1.15). The proof is completed. □

When the matrix triple (AB^T, I, C) is R-controllable, that is rank $[I - sAB^T - C] = n$, for any $s \in R$, the approach of theorem 8.2 can be readily employed to solve the matrix equation (8.0.4) without use of Smith normal form reduction. This is summarized as the following corollary.

Corollary 8.1. Let the matrices $A \in \Re^{n\times p}$, $B \in \Re^{n\times p}$ and $C \in \Re^{n\times r}$ be given, suppose that $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda\{B^T A\} = \phi$ and (AB^T, I, C) is R-controllable. Let the polynomial matrix $H(s) = \sum_{i=0}^{t} H_i s^i \in \Re^{r\times n}$ satisfy

$$\sum_{i=0}^{t}(AB^T)^{t-i}CH_i = -(AB^T)^t. \tag{8.1.20}$$

Then all the solution to the matrix equation (8.0.4) can be characterized by

$$\begin{cases} X = & Ctr(AB^T, C)S_n(f_{(I,AB^T)}(s)I_r)\,Obs_n(A^TB, Z) \\ & + A[Ctr(AB^T, C)S_n(f_{(I,AB^T)}(s)I_r)\,Obs_n(A^TB, Z)]^TB \\ & + [Ctr(AB^T, C)S_t(H(s)) + Ctr(AB^T, I_n)]\,Obs_t(A^TB, R) \\ & + A\{[Ctr(AB^T, C)S_t(H(s)) + Ctr_t(AB^T, I_n)]\,Obs_t(A^TB, R)\}^TB, \\ Y = & Zf_{(I,AB^T)}(A^TB) + \sum\limits_{i=0}^{t} H_iR(A^TB)^i, \end{cases}$$

where Z is an arbitrary chosen parameter matrix.

When (AB^T, I, C) is R-controllable, the polynomial $\Delta(s)$ can be chosen to be 1. In this case, $[\Delta(A^TB)]^{-1} = I_n$, $S_t(\Delta(s)I_n) = I$, and the matrix equation (8.1.11) becomes the the matrix equation (8.1.20). Denote $[H_t^T\ H_{t-1}^T\ \cdots\ H_0^T]^T$, this equation can be rewritten as

$$Ctr_{t+1}(AB^T, C)H_{[t]} = -(AB^T)^t. \tag{8.1.21}$$

If the number t is sufficiently large, one has

$$\operatorname{rank}[Ctr_{t+1}(AB^T, C)] = \operatorname{rank}[Ctr_{t+1}(AB^T, C) - (AB^T)^t].$$

In this case, the matrices H_i, $i \in [1, t]$, can be obtained from (8.1.21) by using some numerically reliable approaches such as QR decomposition approach and SVD approach. Furthermore, when the polynomial $\Delta(s)$ is selected to be s^t, then $S_t(\Delta(s)I_n) = 0$. Similar to the above arguments, the following corollary can be easily obtained.

Corollary 8.2. Let $A \in \Re^{n\times p}$, $B \in \Re^{n\times p}$ and $C \in \Re^{n\times r}$ be given, suppose that $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda\{B^TA\} = \phi$ and (AB^T, I, C) is R-controllable. If the polynomial matrix $H(s) = \sum_{i=0}^{t} H_is^i \in \Re^{r\times n}$ satisfy

$$\sum_{i=0}^{t}(AB^T)^{t-i}CH_i = -I.$$

Then all the solutions to the matrix equation (8.0.4) can be stated as

$$\begin{cases} X = & Ctr(AB^T, C)S_n(f_{(I,AB^T)}(s)I_r)\,Obs_n(A^TB, Z) \\ & + A[Ctr(AB^T, C)S_n(f_{(I,AB^T)}(s)I_r)\,Obs_n(A^TB, Z)]^TB \\ & + Ctr(AB^T, C)S_t(H(s))\,Obs_t(A^TB, R(A^TB)^{-t}) \\ & + A[Ctr(AB^T, C)S_t(H(s))\,Obs_t(A^TB, R(A^TB)^{-t}]^TB, \\ Y = & Zf_{(I,AB^T)}(A^TB) + \sum\limits_{i=0}^{t} H_iR(A^TB)^i, \end{cases}$$

where Z is an arbitrary chosen parameter matrix.

8.2 The Second Approach

In this section, we propose an alternative approach to solve the Yakubovich-transpose matrix equation (8.0.1). Before proceeding, we need the following lemma on the matrix equation

$$X_0 - AX_0^T B = R, \tag{8.2.1}$$

where $A \in R^{n\times p}$, $B \in \Re^{n\times p}$ and $R \in \Re^{n\times p}$ are known matrices, and X_0 is the matrix to be determined.

Lemma 8.3. Let the matrices $A \in \Re^{n\times p}$, $B \in \Re^{n\times p}$ and $R \in \Re^{n\times p}$ be given, suppose that $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda\{B^T A\} = \phi$, and let

$$f_{(I,AB^T)}(s) = \det(I - sAB^T) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0, \alpha_0 = 1,$$

and

$$\operatorname{adj}(I - sAB^T) = \sum_{i=0}^{n-1} R_i s^i. \tag{8.2.2}$$

Then the unique solution to the matrix equation (8.2.1) can be characterized by

$$X_0 = \Big[\sum_{i=0}^{n-1} R_i R(A^T B)^i + \sum_{i=0}^{n-1} R_i AR^T B(A^T B)^i\Big] \times [f_{(I,AB^T)}(A^T B)]^{-1}. \tag{8.2.3}$$

Proof. According to the theorem 8.2 in the reference[80], we have

$$\begin{aligned} X_0 =\Big[&\sum_{i=0}^{n-1}\sum_{k=0}^{i} \alpha_k (AB^T)^{i-k} R(A^T B)^i \\ &+ \sum_{i=0}^{n-1}\sum_{k=0}^{i} \alpha_k (AB^T)^{i-k} AR^T B(A^T B)^i\Big]\Big[f_{(I,AB^T)}(A^T B)\Big]^{-1}. \end{aligned}$$

By the relation (8.1.9), we have

$$R_i = \sum_{k=0}^{i} \alpha_k (AB^T)^{i-k}, \alpha_0 = 1, i = 1, 2, \ldots, n-1.$$

So we can obtain the following conclusion

$$X_0 = \left[\sum_{i=0}^{n-1} R_i R(A^T B)^i + \sum_{i=0}^{n-1} R_i AR^T B(A^T B)^i\right] \times \Big[f_{(I,AB^T)}(A^T B)\Big]^{-1}.$$

Thus, the proof has been completed. □

Suppose that a solution to the nonhomogeneous Yakubovich-transpose matrix equation (8.0.1) can be expressed as $X = T + \widetilde{X}$, $Y = \widetilde{Y}$ with $(\widetilde{X}, \widetilde{Y})$ being a solution to matrix equation (8.0.2). Then one has

$$\begin{aligned}&(T+\widetilde{X}) - A(T+\widetilde{X})^T B - C\widetilde{Y} - R\\&\quad = \widetilde{X} - A\widetilde{X}^T B + T - AT^T B - C\widetilde{Y} - R = T - AT^T B - R.\end{aligned}$$

It follows from this relation that $X = T + \widetilde{X}$, $Y = \widetilde{Y}$ is a solution to (8.0.2) if $X_0 = T$ is a solution to (8.2.1). Combining this fact with the solution (8.1.10) to the matrix equation (8.0.2), now we have the following conclusion on the solution to the Yakubovich-transpose matrix equation (8.0.1). Different from theorems 8.1 and 8.2, one does not need to solve the polynomial matrices $H(s)$ and $L(s)$, and polynomial $\Delta(s)$ when applying theorem 8.3 to obtain general solutions to the Yakubovich-transpose matrix equation (8.2.1).

Theorem 8.3. Given matrices $A \in \mathfrak{R}^{n\times p}$, $B \in \mathfrak{R}^{n\times p}$ and $C \in \mathfrak{R}^{n\times r}$, suppose that $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda\{B^T A\} = \phi$, and (8.2.2) holds. Then all solutions to the nonhomogeneous Yakubovich-transpose matrix equation (8.0.4) can be expressed as

$$\begin{cases} X = \sum\limits_{i=0}^{n-1} R_i CZ(A^T B)^i + \sum\limits_{i=0}^{n-1} A(B^T A)^i (R_i CZ)^T B \\ \qquad + \left[\sum\limits_{i=0}^{n-1} R_i R(A^T B)^i + \sum\limits_{i=0}^{n-1} R_i AR^T B(A^T B)^i\right] \times \left[f_{(I,AB^T)}(A^T B)\right]^{-1}, \\ Y = Zf_{(I,AB^T)}(A^T B), \end{cases}$$

where $Z \in \mathfrak{R}^{r\times p}$ is an arbitrary chosen free parameter matrix.

In addition, it is easily known from theorem 8.3 that

$$\begin{cases} X = \left[\sum\limits_{i=0}^{n-1} R_i R(A^T B)^i + \sum\limits_{i=0}^{n-1} R_i AR^T B(A^T B)^i\right] \times \left[f_{(I,AB^T)}(A^T B)\right]^{-1}, \\ Y = 0, \end{cases}$$

is also a special solution to the matrix equation (8.0.4).

Similar to derivation in section 9.1, we can also obtain some equivalent forms of the solution in theorem 8.3.

Corollary 8.3. Let the matrices $A \in \mathfrak{R}^{n\times p}$, $B \in \mathfrak{R}^{n\times p}$ and $C \in \mathfrak{R}^{n\times r}$ be given, suppose that $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda\{B^T A\} = \phi$, and (8.2.2) holds. Then all solutions to the nonhomogeneous Yakubovich matrix equation (8.0.4) can be characterized by

$$\begin{cases} X = \sum\limits_{i=0}^{n-1} R_i[CZ + R[f_{(I,AB^T)}(A^T B)]^{-1}](A^T B)^i + \sum\limits_{i=0}^{n-1} \{(AB^T)^i A(R_i CZ)^T \\ \qquad + R_i A[(AB^T)^i R]^T [f_{(I,AB^T)}(A^T B)]^{-1}\}B, \\ Y = Zf_{(I,AB^T)}(A^T B), \end{cases}$$

where $Z \in \mathfrak{R}^{r\times p}$ is an arbitrary chosen free parameter matrix.

Corollary 8.4. Let the matrices $A \in \Re^{n\times p}$, $B \in \Re^{n\times p}$ and $C \in \Re^{n\times r}$ be given, suppose that $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda\{B^TA\} = \phi$, and (8.2.2) holds. Then all solutions to the nonhomogeneous Yakubovich matrix equation (8.0.4) can be characterized by

$$\begin{cases} X = A\Big([Ctr(AB^T, [C\ R])]^T S_n(f_{(I,AB^T)}(s)I_{r+p}) \\ \qquad Obs_n\left(B^TA, \begin{bmatrix} ZB^T \\ R[f_{(I,AB^T)}(A^TB)]^{-1}B \end{bmatrix}\right)\Big)^T \\ \qquad + Ctr(AB^T, [CR])S_n(f_{(I,AB^T)}(s)I_{r+p}) \\ \qquad Obs_n\left(A^TB, \begin{bmatrix} Z \\ R[f_{(I,AB^T)}(A^TB)]^{-1} \end{bmatrix}\right), \\ Y = Zf_{(I,AB^T)}(A^TB), \end{cases}$$

where $Z \in \Re^{r\times p}$ is an arbitrary chosen free parameter matrix.

8.3 Illustrative Example

In this section, we give an example to obtain the solution of the nonhomogeneous Yakubovich-transpose matrix equation in the form of (8.0.4).

Example 8.1. Consider the nonhomogeneous Yakubovich-transpose matrix equation in the form of (8.0.1) with the following parameters:

$$A = \begin{pmatrix} -2 & -4 & -3 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, R = \begin{pmatrix} 30 & 22 & -4 \\ -1 & -1 & 6 \\ -11 & -3 & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, C = \begin{pmatrix} 30.5530 & 13.7765 & 6.9924 \\ -7.3409 & -0.3371 & 1.8106 \\ -1.1667 & 0.4167 & 0.8333 \end{pmatrix}.$$

It is easy to check that $\{s|\det(I - sAB^T) = 0\}\bigcap\lambda(B^TA) = \phi$. By simple computations, we have

$$\det(I - sAB^T) = s^3 - 3s^2 + 3s + 1,$$

$$f_{(I,AB^T)}(A^TB) = \begin{pmatrix} -18 & 6 & -30 \\ -72 & 24 & -108 \\ -78 & 24 & -114 \end{pmatrix},$$

$$\left[f_{(I,AB^T)}(A^TB)\right]^{-1} = \begin{pmatrix} 0.3333 & 0.0833 & -0.1667 \\ -0.5000 & 0.6667 & -0.5000 \\ -0.3333 & 0.0833 & 0 \end{pmatrix}.$$

In addition, we have

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -2 & -8 & -1 \\ 1 & 6 & 1 \\ 0 & -2 & 2 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -1 & -6 & -5 \\ 1 & 5 & 4 \\ -2 & -10 & -7 \end{pmatrix}.$$

Choose $Z = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 0 & -4 \\ 0 & 3 & 1 \end{pmatrix}$. So by theorem 8.3 we have

$$X = \begin{pmatrix} 3.0000 & 12.0000 & -3.0000 \\ 1.0000 & 9.0000 & 13.0000 \\ 5.0000 & 5.0000 & 1.0000 \end{pmatrix},$$

$$Y = \begin{pmatrix} -84.0000 & 30.0000 & -132.0000 \\ 348.0000 & -108.0000 & 516.0000 \\ -294.0000 & 96.0000 & -438.0000 \end{pmatrix}.$$

8.4 Conclusions

In this chapter, we have proposed parametric solutions to the nonhomogeneous Yakubovich-transpose matrix equation. These solutions can provide all the degrees of freedom, which are represented by an arbitrarily chosen parameter matrix Z. All the coefficient matrices are not restricted to be in any canonical form. The matrix B explicitly appears in the solutions, and thus can be unknown *a priori*. Therefore, the matrix B can also be viewed as some free design parameters in some problems related to the matrix equation (8.0.4). Meanwhile, an equivalent form of the solutions to the nonhomogeneous Yakubovich-transpose matrix equation has been expressed in terms of the controllability matrix associated with A, B, and C, and observability matrix associated with A, B and the free parameter matrix Z. Such a feature may bring greater convenience and advantages to some problems related to the nonhomogeneous Yakubovich-transpose matrix equation. Moreover, by specifying the obtained solutions, some expressions can also be given for the solutions to the normal transpose matrix equations. From the discussion in this chapter, one can observe that the solutions to the nonhomogeneous Yakubovich-transpose matrix equation are crucial as the theoretical basis of the development of many kinds of other matrix equations, and are deserved further investigation in the future. In addition, as the theoretical generalization of the well-known Karm-Yakubovich-transpose matrix equation, it may be helpful for future control applications.

Chapter 9

Explicit Solutions to the Matrix Equations $XB - AX = CY$ and $XB - A\widehat{X} = CY$

In this chapter, we are concerned with the explicit solution to the generalized Sylvester matrix equation in the real field and the generalized quaternion $\jmath$-conjugate matrix equation over the quaternion field. In order to obtain the explicit solution of the generalized Sylvester conjugate matrix equation, we present a polynomial matrix algorithm. It is also shown that the explicit solution to the considered matrix equation is expressed as the coefficient matrices A, B, C, the free parameter matrix Z and the coefficient matrix of the polynomial matrix $\operatorname{adj}(sI_{2n} - A\overline{A})$. Moreover, we study the generalized Sylvester quaternion $\jmath$-conjugate matrix equation $XB - A\widehat{X} = CY$ based on the idea of the references[6,149,160,184,187,188] and provide the closed-form solutions to this quaternion $\jmath$-conjugate matrix equation by applying the real representation of a quaternion matrix. Compared to the complex representation method[184], the real representation method does not require the coefficient matrix A to be any special case. As the special case of the generalized Sylvester quaternion $\jmath$-conjugate matrix equation, the Sylvester quaternion $\jmath$-conjugate matrix equation $XB - A\widehat{X} = C$ is also investigated. The explicit solutions to this kind of matrix equation have been established.

9.1 Real Matrix Equation $XB - AX = CY$

We consider the following generalized Sylvester matrix equation in real field

$$XB - AX = CY. \tag{9.1.1}$$

Theorem 9.1. Let $A \in \Re^{n\times n}$, $B \in \Re^{p\times p}$ and $C \in \Re^{n\times r}$ be given real matrices. Suppose $\lambda(A) \cap \lambda(B) = \phi$, the following relations

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$$f_A(s) = \det(sI - A) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0, \alpha_n = 1,$$

and

$$\mathrm{adj}(sI - A) = R_{n-1} s^{n-1} + \cdots + R_1 s + R_0,$$

hold true. Then all the solutions to the generalized Sylvester matrix equation (9.1.1) can be established as

$$\begin{cases} X = \sum\limits_{i=0}^{n-1} R_i CZB^i, \\ Y = Zf_A(B), \end{cases} \tag{9.1.2}$$

where the matrix $Z \in R^{r\times p}$ is an arbitrary matrix over real field.

Proof. First of all, we prove matrices X and Y given in (9.1.2) are the solutions to the matrix equation (9.1.1). By the direct calculation we can obtain

$$\begin{aligned} XB - AX &= \sum_{i=0}^{n-1} R_i CZB^{i+1} - A\sum_{i=0}^{n-1} R_i CZB^i \\ &= R_{n-1} CZB^n + \sum_{i=1}^{n-1}(R_{i-1} - AR_i) CZB^i - AR_0 CZ. \end{aligned} \tag{9.1.3}$$

According to the relation $(sI - A)\mathrm{adj}(sI - A) = I\det(sI - A)$, it is easy to derive the following relations

$$\begin{cases} -AR_0 = \alpha_0 I, \\ R_{i-1} - AR_i = \alpha_i I, \quad i = 1, 2, \ldots, n-1, \\ R_{n-1} = \alpha_n I. \end{cases} \tag{9.1.4}$$

So one has

$$R_{n-1} CZB^n + \sum_{i=1}^{n-1}(R_{i-1} - AR_i) CZB^i - AR_0 CZ = CZ\sum_{i=0}^{n} \alpha_i B^i = CZf_A(B) = CY.$$

Therefore, matrices X and Y given in (9.1.2) satisfy the matrix equation (9.1.1).

Next, we prove the completeness of the solution (9.1.2). It follows from theorem 6 of[31] that there are rp degrees of freedom in the solution of the matrix equation (9.1.1), while solution (9.1.2) has exactly rp parameters represented by the elements of the free matrix Z. So we only need to show that all the parameters in the matrix Z contribute to the solution. For all this, it suffices to show that the mapping $Z \to (X, Y)$ defined by (9.1.3) is injective. This is true since $f_A(B)$ is nonsingular under the condition of $\lambda(A) \cap \lambda(B) = \phi$. Thus, the proof has been completed. □

Remark 1. This result is the same as corollary 2 in[150]. In[150], the authors presented the solution to the generalized Sylvester matrix equation (9.1.1) in terms of the Krylov matrix, a symmetric operator matrix, and the generalized observability matrix by applying matrix polynomial and linear solution space. Corollary 2 in[150] is an equivalent form of the presented solution. However, here we establish the solution by applying the characteristic polynomial $sI - A$ of coefficient matrix A and the coefficient matrix of the characteristic polynomial $sI - A$.

In reference[186], we can find the following well-known generalized Faddeev-Leverrier algorithm:

$$\begin{cases} R_{n-k} = R_{n-k+1}A + \alpha_{n-k+1}I_n, & R_n = 0, k = 1, 2, \ldots, n-1, \\ \alpha_{n-k} = \dfrac{\operatorname{trace}(R_{n-k}A)}{k}, & \alpha_n = 1, k = 1, 2, \ldots, n-1, \end{cases} \tag{9.1.5}$$

where $\alpha_i, i = 0, 1, 2, \ldots, n-1$, are the coefficients of the characteristic polynomial of the matrix A, and $R_i, i = 0, 1, \ldots, n-1$, are the coefficient matrices of the adjoint matrix $\operatorname{adj}(sI_n - A)$.

Theorem 9.2. Let $A \in \Re^{n\times n}$, $B \in \Re^{p\times p}$ and $C \in \Re^{r\times p}$ be given matrices, suppose

$$f_A(s) = \det(sI - A) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0, \alpha_n = 1.$$

Then matrices X and Y given by (9.1.2) have the following equivalent form

$$\begin{cases} X = \sum\limits_{i=0}^{n-1} \sum\limits_{k=i+1}^{n} \alpha_k A^{k-i-1} CZB^i, \\ Y = Zf_A(B). \end{cases} \tag{9.1.6}$$

Proof. Due to the relation (9.1.5), it is easy to obtain

$$\begin{cases} R_0 = \alpha_1 I + \alpha_2 A + \cdots + \alpha_{n-1}A^{n-2} + A^{n-1}, \\ \cdots \\ R_{n-2} = \alpha_{n-1}I + A, \\ R_{n-1} = I. \end{cases}$$

This relation can be compactly expressed as

$$R_i = \sum_{k=i+1}^{n} \alpha_k A^{k-i-1}, \alpha_n = 1, i = 1, 2, \ldots, n-1.$$

Substituting the above relation into the expression of X in (9.1.6) and recording the sum, we have

$$X = \sum_{i=0}^{n-1} R_i CZB^i = \sum_{i=0}^{n-1} \left(\sum_{k=i+1}^{n} \alpha_k A^{k-i-1} \right) CZB^i = \sum_{i=0}^{n-1} \sum_{k=i+1}^{n} \alpha_k A^{k-i-1} CZB^i.$$

Combining this with theorem 9.1 we can prove the conclusion. □

Remark 2. The above conclusion is essentially the same as theorem 9.2 in[149]. In[149], the authors provided the solution to the generalized Sylvester matrix equation (9.1.1) with the column vector form of the unknown matrices X and Y by means of the matrix polynomial and Faddeev-Leverrier algorithm. In the current paper, we present the equivalent form of theorem 9.1 using the well-known generalized Faddeev-Leverrier algorithm.

9.2 Quaternion-j-Conjugate Matrix Equation $XB - A\widehat{X} = CY$

9.2.1 Real Representation of a Quaternion Matrix

For any quaternion matrix $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathfrak{Q}^{m\times n}$, $A_l \in \Re^{m\times n}$ $(l = 1, 2, 3, 4)$, the real representation matrix of quaternion matrix A can be defined as

$$A_\sigma = \begin{pmatrix} A_1 & A_2 & -A_3 & A_4 \\ A_2 & -A_1 & -A_4 & -A_3 \\ A_3 & -A_4 & A_1 & A_2 \\ A_4 & A_3 & A_2 & -A_1 \end{pmatrix} \in \Re^{4m\times 4n}. \tag{9.2.1}$$

For an $m \times n$ quaternion matrix A, we define $A_\sigma^t = (A_\sigma)^t$. In addition, if we let

$$P_t = \begin{pmatrix} I_t & 0 & 0 & 0 \\ 0 & -I_t & 0 & 0 \\ 0 & 0 & I_t & 0 \\ 0 & 0 & 0 & -I_t \end{pmatrix}, Q_t = \begin{pmatrix} 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t \\ 0 & 0 & -I_t & 0 \end{pmatrix},$$

$$S_t = \begin{pmatrix} 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \\ 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{pmatrix}, R_t = \begin{pmatrix} 0 & 0 & I_t & 0 \\ 0 & 0 & 0 & I_t \\ -I_t & 0 & 0 & 0 \\ 0 & -I_t & 0 & 0 \end{pmatrix},$$

in which I_t is a $t \times t$ identity matrix, then P_t, Q_t, S_t, R_t are unitary matrices.

For the expression (9.2.1), we can see that $\sigma : A \to A_\sigma$ is a mapping from the quaternion matrix A to its real representation matrix A_σ for any quaternion matrix A. In this case, for a real matrix $A \in \Re^{4n\times 4n}$, the invertible mapping σ^{-1} can be stated as: $\sigma^{-1} : A_{\sigma^{-1}} = B$, in which $B \in Q^{n\times n}$ is a quaternion matrix. We observe that the mapping σ is an injective homomorphism. The real representation matrix has the following properties, which are given in[188].

Proposition 9.1. Let $A, B \in \mathfrak{Q}^{m\times n}$, $C \in \mathfrak{Q}^{n\times s}$, $a \in R$. Then

(1) $(A+B)_\sigma = A_\sigma + B_\sigma$, $(aA)_\sigma = aA_\sigma$, $(AC)_\sigma = A_\sigma P_n C_\sigma = A_\sigma(\widehat{C})_\sigma P_s$;

(2) $A = B \Longleftrightarrow A_\sigma = B_\sigma$;

(3) $Q_m^{-1} A_\sigma Q_n = -A_\sigma$, $R_m^{-1} A_\sigma R_n = A_\sigma$, $S_m^{-1} A_\sigma S_n = -A_\sigma$, $P_m^{-1} A_\sigma P_n = (\widehat{A})_\sigma$;

(4) The quaternion matrix A is nonsingular if and only if A_σ is nonsingular, and the quaternion matrix A is an unitary matrix if and only if A_σ is an orthogonal matrix;

(5) If $A \in \mathfrak{Q}^{m\times m}$, then $A_\sigma^{2k} = ((A\widehat{A})^k)_\sigma P_m$;

(6) $A \in \mathfrak{Q}^{m\times m}, B \in \mathfrak{Q}^{n\times n}$, $C \in \mathfrak{Q}^{m\times n}$ and $k + l$ is even, then

$$A_\sigma^k C_\sigma B_\sigma^l = \begin{cases} ((A\widehat{A})^s(A\widehat{C}B)(\widehat{B}B)^t)_\sigma, & k = 2s+1, l = 2t+1, \\ ((A\widehat{A})^s C(\widehat{B}B)^t)_\sigma, & k = 2s, l = 2t. \end{cases}$$

Proposition 9.2. If λ is a characteristic value of A_σ, then so are $\pm\lambda$ and $\pm\overline{\lambda}$.

For any $A \in \mathfrak{Q}^{m\times m}$, let the characteristic polynomial of the real representation matrix A_σ be $f_{A_\sigma}(\lambda) = \det(\lambda I_{4m} - A_\sigma) = \sum_{k=0}^{2m} a_{2k}\lambda^{2k}$, and define $h_{A_\sigma}(\lambda) = \lambda^{4m} f_{A_\sigma}(\lambda^{-1}) = \sum_{k=0}^{2m} a_{2k}\lambda^{2(2m-k)}$. So by propositions 9.1 and 9.2 we have the following proposition 9.3.

Proposition 9.3. Let $A \in Q^{m\times m}, B \in \mathfrak{Q}^{n\times n}$. Then

(1) $f_{A_\sigma}(\lambda)$ is a real polynomial, and $f_{A_\sigma}(\lambda) = \sum_{k=0}^{2m} a_{2k}\lambda^{2k}$;
(2) $h_{A_\sigma}(\lambda)$ is a real polynomial, and $h_{A_\sigma}(\lambda) = \sum_{k=0}^{2m} a_{2k}\lambda^{2(2m-k)}$;
(3) $h_{A_\sigma}(B_\sigma) = (g_{A_\sigma}(B\widehat{B}))_\sigma P_n, f_{A_\sigma}(B_\sigma) = (p_{A_\sigma}(B\widehat{B}))_\sigma P_n$, in which $g_{A_\sigma}(\lambda) = \sum_{k=0}^{2m} a_{2k}\ \lambda^{2m-k}$, $p_{A_\sigma}(\lambda) = \sum_{k=0}^{2m} a_{2k}\lambda^k$ are real polynomials.

Proof. By proposition 9.2, we easily know that a_k is a real number, and $a_{2k+1} = 0$. For any k, by proposition 9.1, we have $B_\sigma^{2k} = ((B\widehat{B})^k)_\sigma P_n$. Thus, we can obtain the result (3). ☐

Furthermore, we have the following conclusion on the relation of quaternion matrix and quaternion j-conjugate matrix.

Proposition 9.4. Let $A \in \mathfrak{Q}^{m\times m}, B \in \mathfrak{Q}^{m\times m}$. Then

(1) $A = B \Longleftrightarrow \widehat{A} = \widehat{B}$;
(2) $\widehat{A} = B \Longleftrightarrow A = \widehat{B}$;
(3) $\widehat{\widehat{A}} = A$;
(4) $\widehat{AB} = \widehat{A}\,\widehat{B}$.

Proof. By the definition of quaternion j-conjugate matrix $\widehat{A}$, items (1), (2) and (3) can be proved. Now we prove item (4). Let $A = A_1 + A_2 i + A_3 j + A_4 k$, $B = B_1 + B_2 i + B_3 j + B_4 k$. By calculation we have

$$\begin{aligned} AB =& (A_1B_1 - A_2B_2 - A_3B_3 - A_4B_4) + (A_1B_2 + A_2B_1 + A_3B_4 - A_4B_3)i \\ &+ (A_1B_3 + A_3B_1 - A_2B_4 + A_4B_2)j + (A_4B_1 + A_1B_4 + A_2B_3 - A_3B_2)k, \end{aligned}$$

$$\begin{aligned} \widehat{AB} =& (A_1B_1 - A_2B_2 - A_3B_3 - A_4B_4) - (A_1B_2 + A_2B_1 + A_3B_4 - A_4B_3)i \\ &+ (A_1B_3 + A_3B_1 - A_2B_4 + A_4B_2)j - (A_4B_1 + A_1B_4 + A_2B_3 - A_3B_2)k, \end{aligned}$$

and

$$\begin{aligned} \widehat{A}\,\widehat{B} =& (A_1B_1 - A_2B_2 - A_3B_3 - A_4B_4) + (-A_1B_2 - A_2B_1 - A_3B_4 + A_4B_3)i \\ &+ (A_1B_3 + A_3B_1 - A_2B_4 + A_4B_2)j + (-A_4B_1 - A_1B_4 - A_2B_3 + A_3B_2)k. \end{aligned}$$

Hence we have $\widehat{AB} = \widehat{A}\,\widehat{B}$. □

In the real field, for two square matrices, if there exists a real number λ satisfies $\det(A + \lambda B) \neq 0$, then the real matrix pair (A, B) is called real regular matrix pair. Similarly, we have the following definition on the quaternion regular matrix pair.

Definition 9.1. Let $A, B \in \mathfrak{Q}^{m\times m}$. If there exists $\lambda \in \Re$ such that

$$\det(A_\sigma + \lambda B_\sigma) \neq 0,$$

then the quaternion matrix pencil (A, B) is called a quaternion regular matrix pair.

According to definition 9.1, it is easy to obtain the following lemma.

Lemma 9.1. Quaternion matrix pair (A, B) is a quaternion regular matrix pair if and only if its real representation matrix pair (A_σ, B_σ) is a real regular matrix pair.

9.2.2 Solutions to the Quaternion j-Conjugate Matrix Equation $XB - A\widehat{X} = CY$

In this subsection, we will discuss the solution of the following quaternion j-conjugate matrix equation

$$XB - A\widehat{X} = CY, \tag{9.2.2}$$

in which $A \in \mathfrak{Q}^{n\times n}$, $B \in \mathfrak{Q}^{p\times p}$ and $C \in \mathfrak{Q}^{n\times r}$ are known quaternion matrices, $X \in \mathfrak{Q}^{n\times p}$ and $Y \in \mathfrak{Q}^{r\times p}$ are unknown quaternion matrices.

Firstly, we define the real representation of the generalized Sylvester quaternion j-conjugate matrix equation (9.2.2) by

$$V(\hat{B})_\sigma - A_\sigma V = C_\sigma W. \tag{9.2.3}$$

Due to propositions 9.1 and 9.4, the generalized Sylvester quaternion matrix equation (9.2.2) is equivalent to the following matrix equation

$$(XB - A\widehat{X})_\sigma = X_\sigma P_n B_\sigma - A_\sigma X_\sigma P_p.$$

Post-multiplying the two sides of the above quaternion matrix equation by P_p, the matrix equation (9.2.2) can be converted into

$$X_\sigma(\hat{B})_\sigma - A_\sigma X_\sigma = C_\sigma(\hat{Y})_\sigma.$$

In this case, we have the following conclusion.

Proposition 9.5. Let $A \in \mathfrak{Q}^{n\times n}$, $B \in \mathfrak{Q}^{p\times p}$ and $C \in \mathfrak{Q}^{n\times r}$ be given quaternion matrices. Then the quaternion matrix equation (9.2.3) has a solution (X, Y) if and only if the real representation matrix equation (9.2.3) has a solution $(V, W) = (X_\sigma, (\hat{Y})_\sigma)$.

Theorem 9.3. Suppose $A \in \mathfrak{Q}^{n\times n}$, $B \in \mathfrak{Q}^{p\times p}$, $C \in \mathfrak{Q}^{n\times r}$ be given quaternion matrices. Then the quaternion matrix equation (9.2.3) has a solution (X, Y) if and only if the real representation matrix equation (9.2.3) has a solution (V, W). Furthermore, if (V, W) is a solution to (9.2.3), then the following quaternion matrices are solutions to (9.2.3):

$$\begin{cases} X = \dfrac{1}{16}[I_n \ iI_n \ jI_n \ kI_n] \times (V - Q_n^{-1}VQ_p + R_n^{-1}VR_p - S_n^{-1}VS_p)\begin{bmatrix} I_p \\ -iI_p \\ -jI_p \\ -kI_p \end{bmatrix}, \\ Y = \dfrac{1}{16}[I_r \ iI_r \ jI_r \ kI_r] \times (W - Q_n^{-1}WQ_p + R_n^{-1}WR_p - S_n^{-1}WS_p)t\begin{bmatrix} I_p \\ -iI_p \\ -jI_p \\ -kI_p \end{bmatrix}. \end{cases} \tag{9.2.4}$$

Proof. By (3) of proposition 9.1, the quaternion matrix equation (9.2.2) is equivalent to

$$VR_p^{-1}(\widehat{B})_\sigma R_p - R_n^{-1}A_\sigma R_n V = R_n^{-1}C_\sigma R_r W. \tag{9.2.5}$$

Post-multiplying both sides of the matrix equation (9.2.5) by R_p^{-1}, we can obtain

$$VR_p^{-1}(\hat{B})_\sigma - R_n^{-1}A_\sigma R_n VR_p^{-1} = R_n^{-1}C_\sigma R_r WR_p^{-1}. \tag{9.2.6}$$

Pre-multiplying both sides of the matrix equation (9.2.6) by R_n, we have

$$R_n VR_p^{-1}(\hat{B})_\sigma - A_\sigma R_n VR_p^{-1} = C_\sigma R_r WR_p^{-1}. \tag{9.2.7}$$

Noting $R_p^{-1} = -R_p$, we give

$$R_n^{-1}VR_p(\hat{B})_\sigma - A_\sigma R_n^{-1}VR_p = C_\sigma R_r^{-1}WR_p.$$

This shows that if (V, W) is a real solution of matrix equation (9.2.3), then $(R_n^{-1}VR_p, R_r^{-1}WR_p)$ is also a real solution of quaternion matrix equation (9.2.3). In addition, according to (3) of proposition 9.1, the quaternion matrix equation (9.2.3) is also equivalent to

$$VQ_p^{-1}(\hat{B})_\sigma Q_p - Q_n^{-1}A_\sigma Q_n V = Q_n^{-1}C_\sigma Q_r W. \tag{9.2.8}$$

Post-multiplying both sides of the quaternion matrix equation (9.2.8) by Q_p^{-1}, we have

$$VQ_p^{-1}(\hat{B})_\sigma - Q_n^{-1}A_\sigma Q_n VQ_p^{-1} = Q_n^{-1}C_\sigma Q_r WQ_p^{-1}. \tag{9.2.9}$$

Pre-multiplying the two sides of the quaternion matrix equation (9.2.9) by Q_n^{-1} and computing $Q_p^{-1} = -Q_p$, give

$$(-Q_n^{-1}VQ_p)(\hat{B})_\sigma - A_\sigma(-Q_n^{-1}VQ_p) = C_\sigma(-Q_r^{-1}WQ_p).$$

This is to say that if (V, W) is a real solution of matrix equation (9.2.3), then $(-Q_n^{-1}VQ_p, -Q_r^{-1}WQ_p)$ is also a real solution of matrix equation (9.2.3). Similarly, we can prove $(-S_n^{-1}VS_p, -S_r^{-1}WS_p)$ is also a real solution of quaternion matrix equation (9.2.3). In this case, the conclusion can be obtained along similar line of the proof of theorem 4.2 in[187]. □

Theorem 9.4. Given quaternion matrices $A \in Q^{n\times n}$, $B \in Q^{p\times p}$ and $C \in Q^{n\times r}$, suppose

$$f_{A_\sigma}(s) = \det(sI_{4n} - A_\sigma) = \sum_{k=0}^{2n} a_{2k}s^{2k},$$

and

$$p_{A_\sigma}(s) = \sum_{k=0}^{2n} a_{2k}s^{k}.$$

Then the quaternion matrices $X \in \mathfrak{Q}^{n\times p}$, $Y \in \mathfrak{Q}^{r\times p}$ can be established as

$$\begin{cases} X = \sum\limits_{k=1}^{2n} \alpha_{2k}\Big(\sum\limits_{s=k-1}^{n-1} (A\hat{A})^{k-s-1}A\hat{C}Z(B\widehat{B})^{s} \\ \qquad + \sum\limits_{s=k-1}^{n-1} (A\widehat{A})^{k-s-1}C\widehat{Z}\widehat{B}(B\widehat{B})^{s}\Big), \\ Y = \hat{Z}p_{A_\sigma}(B\widehat{B}), \end{cases} \tag{9.2.10}$$

in which Z is an arbitrary quaternion matrix.

Proof. If the generalized Sylvester quaternion j-conjugate matrix equation (9.2.2) has solution (X, Y), then the real representation matrix equation (9.2.3) has solution $(V, W) = (X_\sigma, (\hat{Y})_\sigma)$ with the free parameter Z_σ. By theorems 9.2 and 9.3, we have

$$\begin{aligned} X_\sigma &= \sum_{j=0}^{4n-1}\sum_{k=j+1}^{4n} \alpha_k A_\sigma^{k-j-1}C_\sigma Z_\sigma(\hat{B})_\sigma^{j} \\ &= \sum_{j=0}^{2n-1}\sum_{k=j+1}^{2n} \alpha_{2k} A_\sigma^{2k-j-1}C_\sigma Z_\sigma(\hat{B})_\sigma^{j} \\ &= \sum_{k=1}^{2n}\alpha_{2k}\Big[\sum_{j=k-1}^{2n-1} A_\sigma^{2k-j-1}C_\sigma Z_\sigma(\hat{B})_\sigma^{j}\Big] \\ &= \sum_{k=1}^{2n}\alpha_{2k}\Big[\sum_{s=k-1}^{n-1} A_\sigma^{2k-2s-1}C_\sigma Z_\sigma(\hat{B})_\sigma^{2s} + \sum_{s=k}^{n} A_\sigma^{2k-2s}C_\sigma Z_\sigma(\hat{B})_\sigma^{2s-1}\Big] \\ &= \sum_{k=1}^{2n}\alpha_{2k}\Big[\sum_{s=k-1}^{n-1} A_\sigma^{2(k-s-1)}A_\sigma C_\sigma Z_\sigma(\hat{B})_\sigma^{2s} + \sum_{s=k}^{n} A_\sigma^{2(k-s)}C_\sigma Z_\sigma(\hat{B})_\sigma^{2(s-1)+1}\Big] \\ &= \sum_{k=1}^{2n}\alpha_{2k}\Big[\sum_{s=k-1}^{n-1} \Big((A\widehat{A})^{k-s-1}A\hat{C}Z(B\widehat{B})^{s}\Big)_\sigma + \sum_{s=k}^{n}\Big((A\widehat{A})^{k-s}C\widehat{Z}\widehat{B}(B\widehat{B})^{s-1}\Big)_\sigma\Big] \\ &= \sum_{k=1}^{2n}\alpha_{2k}\Big[\sum_{s=k-1}^{n-1} \Big((A\widehat{A})^{k-s-1}A\hat{C}Z(B\widehat{B})^{s}\Big)_\sigma + \sum_{s=k-1}^{n-1}\Big((A\widehat{A})^{k-s-1}C\widehat{Z}\widehat{B}(B\widehat{B})^{s}\Big)_\sigma\Big]. \end{aligned}$$

So we have

$$X = \sum_{k=1}^{2n} \alpha_{2k} \Big[\sum_{s=k-1}^{n-1} (A\widehat{A})^{k-s-1} A\hat{C}Z(B\widehat{B})^{s} + \sum_{s=k-1}^{n-1} (A\widehat{A})^{k-s-1} C\widehat{Z}\widehat{B}(B\widehat{B})^{s} \Big].$$

In addition, according to proposition 9.3, $f_{A_\sigma}(s)$ is a real polynomial and we have the following relation

$$f_{A_\sigma}(B_\sigma) = [p_{A_\sigma}((B\widehat{B})_\sigma)]P_p.$$

Thus, according to proposition 9.1, we have

$$(\hat{Y})_\sigma = Z_\sigma f_{A_\sigma}(B_\sigma) = Z_\sigma [p_{A_\sigma}((B\widehat{B})_\sigma)]P_p = [Zp_{A_\sigma}(\widehat{B}B)]_\sigma.$$

Therefore, from proposition 9.4 we can obtain the following relations

$$\widehat{Y} = Zp_{A_\sigma}(\widehat{B}B),$$

and

$$Y = \hat{Z}p_{A_\sigma}(B\widehat{B}).$$

Thus, we complete the proof of the conclusion. ☐

In the following, based on the main result proposed above, we consider the following Sylvester quaternion j-conjugate matrix equation

$$XB - A\widehat{X} = C. \tag{9.2.11}$$

Corollary 9.1. Given quaternion matrices $A \in \mathfrak{Q}^{n\times n}$, $B \in \mathfrak{Q}^{p\times p}$ and $C \in \mathfrak{Q}^{n\times p}$, let

$$f_{A_\sigma}(s) = \det(sI_{4n} - A_\sigma) = \sum_{k=0}^{2n} a_{2k} s^{2k},$$

and

$$p_{A_\sigma}(s) = \sum_{k=0}^{2n} a_{2k} s^{k}.$$

(1) If X is a solution of the equation (9.2.11), then

$$Xp_{A_\sigma}(\widehat{B}B) = \sum_{k=1}^{2n} \alpha_{2k} \Big[\sum_{s=0}^{k-1} (A\widehat{A})^{s} C\widehat{B}(B\widehat{B})^{k-s-1} + \sum_{s=0}^{k-1} (A\widehat{A})^{s} A\widehat{C}(B\widehat{B})^{k-s-1} \Big]. \tag{9.2.12}$$

(2) If (A, B) is a regular quaternion matrix pair, then the quaternion matrix (9.2.11) has a unique solution

$$X = \sum_{k=1}^{2n} \alpha_{2k} \Big[\sum_{s=0}^{k-1} (A\widehat{A})^s C \widehat{B} (B\widehat{B})^{k-s-1} + \sum_{s=0}^{k-1} (A\widehat{A})^s A \widehat{C} (B\widehat{B})^{k-s-1} \Big] [p_{A_\sigma}(\widehat{B}B)]^{-1}. \tag{9.2.13}$$

Proof. If X is a solution of the equation (9.2.10), then $Y = X_\sigma$ is a solution of the real matrix equation $X_\sigma \hat{B}_\sigma - A_\sigma X_\sigma = C_\sigma P_p$. By theorem 3 in[30] and proposition 9.1, we have

$$X_\sigma f_{A_\sigma}(\hat{B}_\sigma) = \sum_{k=1}^{4n} \sum_{j=0}^{k-1} \alpha_k A_\sigma^j C_\sigma P_p \hat{B}_\sigma^{k-j-1}.$$

From proposition 9.3, $f_{A_\sigma}(s)$ is a real polynomial and $f_{A_\sigma}(\widehat{B}_\sigma) = (p_{A_\sigma}(B\widehat{B}))_\sigma\, P_p$. So using proposition 9.1, we have

$$\begin{aligned}
&[X p_{A_\sigma}(\widehat{B}B)]_\sigma \\
&\quad = X_\sigma [p_{A_\sigma}(B\widehat{B})]_\sigma P_p \\
&\quad = X_\sigma f_{A_\sigma}(\widehat{B}_\sigma) \\
&\quad = \sum_{k=1}^{4n} \sum_{j=0}^{k-1} \alpha_k A_\sigma^j C_\sigma P_p (\widehat{B})_\sigma^{k-j-1} \\
&\quad = \sum_{k=1}^{2n} \sum_{j=0}^{2k-1} \alpha_{2k} A_\sigma^j C_\sigma P_p (\widehat{B})_\sigma^{2k-j-1} \\
&\quad = \sum_{k=1}^{2n} \alpha_{2k} \Big[\sum_{s=0}^{k-1} A_\sigma^{2s} C_\sigma P_p (\widehat{B})_\sigma^{2k-2s-1} + \sum_{s=1}^{k} A_\sigma^{2s-1} C_\sigma P_p (\widehat{B})_\sigma^{2k-2s} \Big] \\
&\quad = \sum_{k=1}^{2n} \alpha_{2k} \Big[\sum_{s=0}^{k-1} A_\sigma^{2s} C_\sigma P_p (\widehat{B})_\sigma^{2(k-s-1)+1} + \sum_{s=1}^{k} A_\sigma^{2s-1} C_\sigma P_p (\widehat{B})_\sigma^{2k-2s} \Big] \\
&\quad = \sum_{k=1}^{2n} \alpha_{2k} \Big[\sum_{s=0}^{k-1} \Big((A\widehat{A})^s C \widehat{B} (B\widehat{B})^{k-s-1} \Big)_\sigma + \sum_{s=1}^{k} \Big((A\widehat{A})^{s-1} A \widehat{C} (B\widehat{B})^{k-s} \Big)_\sigma \Big] \\
&\quad = \sum_{k=1}^{2n} \alpha_{2k} \Big[\sum_{s=0}^{k-1} \Big((A\widehat{A})^s C \widehat{B} (B\widehat{B})^{k-s-1} \Big)_\sigma + \sum_{s=0}^{k-1} \Big((A\widehat{A})^{s} A \widehat{C} (B\widehat{B})^{k-s-1} \Big)_\sigma \Big].
\end{aligned}$$

So we have

$$X p_{A_\sigma}(\widehat{B}B) = \sum_{k=1}^{2n} \alpha_{2k} \Big[\sum_{s=0}^{k-1} (A\widehat{A})^s C \widehat{B} (B\widehat{B})^{k-s-1} + \sum_{s=0}^{k-1} (A\widehat{A})^s A \widehat{C} (B\widehat{B})^{k-s-1} \Big].$$

Thus, the conclusion has been proved. □

In the following, we give an example to obtain the solution of Sylvester quaternion matrix equation $XB - A\widehat{X} = C$.

Example 9.1. Consider the explicit solution of the Sylvester quaterniom matrix equation $XB - A\widehat{X} = C$ with the following parameters:

$$A = \begin{pmatrix} 1 & 2i \\ j & -k \end{pmatrix}, B = \begin{pmatrix} 3+j & 2-k \\ 4i & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 12+4i-6k & 4-4i-10k \\ -2-4i+7j-13k & 1-7i+9j+4k \end{pmatrix}.$$

According to the definition of real representation of a complex matrix, we have

$$A_\sigma = \begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check the quaternion matrix pair (A, B) is regular. By some simple computation, we have

$$f_{A_\sigma}(s) = s^8 - 4s^6 + 6s^4 - 4s^2 + 1,$$

$$p_{A_\sigma}(s) = s^4 - 4s^3 + 6s^2 - 4s + 1,$$

and

$$\alpha_8 = 1, \alpha_6 = -4, \alpha_4 = 6, \alpha_2 = -4, \alpha_0 = 1.$$

It follows from corollary 9.1 that the solution of the Sylvester quaternion matrix equation is

$$X = \begin{pmatrix} 2 & 1-2i \\ 3j+k & 4j \end{pmatrix}.$$

9.3 Conclusions

In this chapter, the real generalized Sylvester matrix equation $XB - AX = CY$ and the quaternion matrix equation $XB - AX = CY$ have been investigated. We have provided a new solution to the real matrix equation $XB - AX = CY$ through a direct method. Based on the solution of this real matrix equation and the real representation of a quaternion matrix, we have established the explicit solutions of the quaternion matrix equation $XB - A\widetilde{X} = CY$. Compared to the existing results[184], there is no requirement for the coefficient matrix A. But in the reference[184], the coefficient matrix A is restricted with a block diagonal complex matrix. In this sense, the results in this chapter can be regarded as the further result of the reference[184]. It is the generalization of existing results. As a special case of the generalized Sylvester quaternion $\jmath$-conjugate matrix equation, Sylvester quaternion $\jmath$-conjugate matrix equation $XB - A\widehat{X} = C$ has also been considered in this paper. The explicit solutions to Sylvester quaternion $\jmath$-conjugate matrix equation have been proposed.

Chapter 10

Explicit Solutions to Linear Transpose Matrix Equation

In this chapter, we consider the explicit solutions to the equations

$$AX + X^T B = C, \tag{10.0.1}$$

and

$$AX + X^T B = CY, \tag{10.0.2}$$

where $A, B, C \in \mathcal{R}^{n\times n}$ are given $n \times n$ real matrices and $X, Y \in \mathcal{R}^{n\times n}$ are unknown $n \times n$ real matrices to be determined. They are named the Sylvester transpose matrix equation and the generalized Sylvester transpose matrix equation, respectively. If $X = X^T$, the equations (10.0.1) and (10.0.2) reduce to the well-known Sylvester (or generalized Sylvester) matrix equation, which has many important applications in control theory. For calculating the solutions to (10.0.1) and (10.0.2), three algorithms are provided in this chapter. The design of a time-varying linear system in control is given.

10.1 Solutions to the Sylvester Transpose Matrix Equation

In this section, the explicit solutions to (10.0.1) are considered and three forms of solutions are constructed. One of the solutions is the well-known Jameson's Theorem. In this sense, we generalize Jameson's Theorem from the Sylvester matrix equation to the Sylvester transpose matrix equation. Firstly, some useful lemmas are summarized below.

Definition 10.1 (*Sylvester sum*[6]). Let $T(s) = \sum_{i=0}^{t} T_i s^i \in \mathcal{C}^{m\times q}[s], F \in \mathcal{C}^{p\times p}$, and $Z \in \mathcal{C}^{q\times p}$, the following matrix sum $\mathrm{Syl}(T(s), F, Z) = \sum_{i=0}^{t} T_i Z F^i$ is called Sylvester sum.

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Definition 10.2 (*Kronecker map*[6]). Consider the polynomial matrix $T(s) = \sum_{i=0}^{t} T_i s^i \in \mathcal{C}^{m\times q}[s]$, for a fixed $F \in \mathcal{C}^{p\times p}$, the map $\Phi[T(s)] = \sum_{i=0}^{t} (F^T)^i \otimes T_i$ is called Φ-Kronecker map.

Lemma 10.1. [6]Let $X(s) \in C^{q\times r}[s]$, $Y(s) \in \mathcal{C}^{r\times m}[s]$, then $\Phi[X(s)Y(s)] = \Phi[X(s)]\Phi[Y(s)]$.

Lemma 10.2. [6]Let $T(s) \in \mathcal{C}^{q\times q}[s]$, then $\Phi[\mathrm{adj}(T(s))]\Phi[T(s)] = \Phi[I_q \det T(s)]$.

10.1.1 The First Case: A or B is Nonsingular

Define

$$f_{(I_n,B^TA^{-1})}(s) := \det(I_n - sB^TA^{-1}) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0, \quad \alpha_0 = 1, \tag{10.1.1}$$

and

$$\mathrm{adj}(I_n - sB^TA^{-1}) := \Upsilon_{n-1}s^{n-1} + \cdots + \Upsilon_1 s + \Upsilon_0. \tag{10.1.2}$$

Theorem 10.1. Let A be nonsingular and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of B^TA^{-1}, thus, we have

(1) Supposed that X is a solution to equation (10.0.1), then

$$Xf_{(I_n,B^TA^{-1})}((B^TA^{-1})^T) = A^{-1}\sum_{i=0}^{n-1}\Upsilon_i(C - (B^TA^{-1}C)^T)((B^TA^{-1})^T)^i. \tag{10.1.3}$$

(2) If $\lambda_i\lambda_j \neq 1$ for all i, j, then equation (10.0.1) has a unique solution

$$X = A^{-1}\left[\sum_{i=0}^{n-1}\Upsilon_i(C - (B^TA^{-1}C)^T)((B^TA^{-1})^T)^i\right] \times f^{-1}_{(I_n,B^TA^{-1})}((B^TA^{-1})^T). \tag{10.1.4}$$

Proof. From lemma 10.1, equation (10.0.1) can be equivalent to

$$\mathrm{Syl}(I_n - sB^TA^{-1}, (B^TA^{-1})^T, AX) = C - (B^TA^{-1}C)^T.$$

By definitions (10.1.1) and (10.1.2), the above relation can be rewritten as

$$\Phi[I_n - sB^TA^{-1}]\mathrm{vec}(AX) = \mathrm{vec}(C - (B^TA^{-1}C)^T).$$

From lemma 10.2, we have

$$\Phi[I_n\det(I_n - sB^TA^{-1})]\mathrm{vec}(AX) = \Phi[\mathrm{adj}(I_n - sB^TA^{-1})]\mathrm{vec}(C - (B^TA^{-1}C)^T).$$

So we get

$$\begin{aligned}&\mathrm{Syl}(I_n\det(I_n - sB^TA^{-1}), (B^TA^{-1})^T, AX)\\&\quad = \mathrm{Syl}(\mathrm{adj}(I_n - sB^TA^{-1}), (B^TA^{-1})^T, C - (B^TA^{-1}C)^T).\end{aligned}$$

By simple computation, one can obtain

$$\mathrm{Syl}(I_n\det(I_n - sB^TA^{-1}), (B^TA^{-1})^T, AX) = AXf_{(I_n,B^TA^{-1})}((B^TA^{-1})^T),$$

and

$$\begin{aligned}&\mathrm{Syl}(\mathrm{adj}(I_n - sB^TA^{-1}), (B^TA^{-1})^T, C - (B^TA^{-1}C)^T)\\&\quad = \sum_{i=0}^{n-1} \Upsilon_i(C - (B^TA^{-1}C)^T)((B^TA^{-1})^T)^i.\end{aligned}$$

Thus, we get

$$Xf_{(I_n,B^TA^{-1})}((B^TA^{-1})^T) = A^{-1}\sum_{i=0}^{n-1} \Upsilon_i(C - (B^TA^{-1}C)^T)((B^TA^{-1})^T)^i.$$

If $\lambda_i\lambda_j \neq 1$ for all i, j, then $f_{(I_n,B^TA^{-1})}((B^TA^{-1})^T)$ is nonsingular. Therefore, equation (10.0.1) has a unique solution and (10.1.4) holds. □

Now, we summarize the well-known generalized Leverrier algorithm[186] as follows:

$$\begin{cases}\Upsilon_k = \Upsilon_{k-1}B^TA^{-1} + \alpha_k I_n, & \Upsilon_0 = I_n, k = 1, 2, \ldots, n,\\ \alpha_k = \dfrac{\mathrm{trace}(\Upsilon_{k-1}B^TA^{-1})}{k}, & \alpha_0 = 1, k = 1, 2, \ldots, n,\end{cases} \tag{10.1.5}$$

where α_i, $i = 0, 1, 2, \ldots, n-1$, are the coefficients of the characteristic polynomial of B^TA^{-1}, and Υ_i, $i = 0, 1, \ldots, n-1$, are the coefficient matrices of $\mathrm{adj}(sI_n - B^TA^{-1})$. Applying (10.1.5), we will provide the following results.

Theorem 10.2. Let A be nonsingular and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of B^TA^{-1}.

(1) Supposed that X is a solution of the equation (10.0.1), then

$$\begin{aligned}&Xf_{(I_n,B^TA^{-1})}((B^TA^{-1})^T)\\&\quad = A^{-1}\sum_{j=0}^{n-1}\sum_{k=0}^{j} \alpha_k(B^TA^{-1})^{j-k}(C - (B^TA^{-1}C)^T)((B^TA^{-1})^T)^j.\end{aligned} \tag{10.1.6}$$

(2) If $\lambda_i\lambda_j \neq 1$ for all i, j, then the unique solution to (10.0.1) is stated as

$$\begin{aligned}X = {}& A^{-1}\Big[\sum_{j=0}^{n-1}\sum_{k=0}^{j} \alpha_k(B^TA^{-1})^{j-k}(C - (B^TA^{-1}C)^T)((B^TA^{-1})^T)^j\Big]\\& \times f^{-1}_{(I_n,B^TA^{-1})}((B^TA^{-1})^T).\end{aligned} \tag{10.1.7}$$

Proof. From (10.1.5), we get

$$\begin{cases} \Upsilon_0 = I_n, \\ \Upsilon_1 = \alpha_1 I_n + B^T A^{-1}, \\ \Upsilon_2 = \alpha_2 I_n + \alpha_1 B^T A^{-1} + (B^T A^{-1})^2, \\ \cdots \\ \Upsilon_{n-1} = \alpha_{n-1} I_n + \alpha_{n-2} B^T A^{-1} + \cdots + (B^T A^{-1})^{n-1}. \end{cases}$$

The above formula can be rewritten as

$$\Upsilon_j = \sum_{k=0}^{j} \alpha_k (B^T A^{-1})^{j-k}, \quad j = 1, 2, \ldots, n-1.$$

Therefore, we have

$$\begin{aligned} &\sum_{j=0}^{n-1} \Upsilon_j (C - (B^T A^{-1} C)^T)((B^T A^{-1})^T)^j \\ &= \sum_{j=0}^{n-1} \left(\sum_{k=0}^{j} \alpha_k (B^T A^{-1})^{j-k} \right) (C - (B^T A^{-1} C)^T)((B^T A^{-1})^T)^j \\ &= \sum_{j=0}^{n-1} \sum_{k=0}^{j} \alpha_k (B^T A^{-1})^{j-k} (C - (B^T A^{-1} C)^T)((B^T A^{-1})^T)^j. \end{aligned}$$

Combining this relation with theorem 10.1, (10.1.6) can be proved. If $\lambda_i \lambda_j \neq 1$ for all i, j, then $f_{(I_n, B^T A^{-1})}((B^T A^{-1})^T)$ is nonsingular. Thus, (10.0.1) has a unique solution. So the desired result follows. □

Remark 1. This theorem is the Jameson's Theorem of the Sylvester transpose matrix equation (10.0.1), which has the same form as theorem 7 in[9].

Theorem 10.3. Supposed that A is nonsingular, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $B^T A^{-1}$ and $C - (B^T A^{-1} C)^T = C_1 C_2$, $C_1 \in \mathcal{R}^{n \times r}$, $C_2 \in \mathcal{R}^{r \times n}$ hold, where $r = \mathrm{rank}(C - (B^T A^{-1} C)^T)$.

(1) Let X be a solution to (10.0.1), then this solution can be stated as

$$\begin{aligned} &X f_{(I_n, B^T A^{-1})}((B^T A^{-1})^T) \\ &\quad = A^{-1} Q_c(B^T A^{-1}, C_1, n) S_r(I, B^T A^{-1}) Q_o((B^T A^{-1})^T, C_2, n). \end{aligned} \tag{10.1.8}$$

(2) If $\lambda_i \lambda_j \neq 1$ for all i, j, then the unique solution to (10.0.1) can be expressed as

$$\begin{aligned} X = {} & A^{-1} \Big[Q_c(B^T A^{-1}, C_1, n) S_r(I, B^T A^{-1}) Q_o((B^T A^{-1})^T, C_2, n) \Big] \\ & \times f^{-1}_{(I_n, B^T A^{-1})}((B^T A^{-1})^T). \end{aligned} \tag{10.1.9}$$

Proof. According to (10.1.5), we have

$$[\,\Upsilon_0 C_1 \quad \Upsilon_1 C_1 \quad \cdots \quad \Upsilon_{n-1} C_1\,] = Q_c(B^TA^{-1}, C_1, n)S_r(I, B^TA^{-1}).$$

Thus, it is easy to obtain

$$\begin{aligned}&\sum_{i=0}^{n-1}\Upsilon_i(C-(B^TA^{-1}C)^T)((B^TA^{-1})^T)^i\\ &\quad = [\,\Upsilon_0 C_1 \quad \Upsilon_1 C_1 \quad \cdots \quad \Upsilon_{n-1} C_1\,]Q_o((B^TA^{-1})^T, C_2, n)\\ &\quad = Q_c(B^TA^{-1}, C_1, n)S_r(I, B^TA^{-1})Q_o((B^TA^{-1})^T, C_2, n).\end{aligned}$$

Combining this relation with theorem 10.2, (10.1.8) can be proved. If $\lambda_i\lambda_j \neq 1$ for all i, j, then $f_{(I_n, B^TA^{-1})}((B^TA^{-1})^T)$ is nonsingular. Therefore, (10.0.1) has a unique solution and (10.1.9) holds. □

Remark 2. From corollary 1 in[24], the sufficient condition for existence of solution to (10.0.1) is that $A^T + C$ and $A^T - C$ are nonsingular under the condition $C^T = C$.

From corollary 2 in[24], the necessary and sufficient conditions for existence of solution to (10.0.1) is that $A^T + C$ is nonsingular under the conditions $C^T = C$ and $(I - P_{21}^T)B_2^*(I - P_{21}) = 0$, where $P_{21} = (A^T - B)^T(A^T + B)$ and $B_2^* = C(A^T + B)^{-1}(A^T - B) + (A^T - B)^T[(A^T + B)^{-1}]^TC$.

Compared with corollaries 1 and 2 in[24], the sufficient condition for existence of solution to (10.0.1) in theorem 10.3 only requests $\lambda\mu \neq 1$ for any $\lambda, \mu \in \sigma(B^TA^{-1})$. This condition is easy to check through Matlab.

Suppose that B is nonsingular. Taking the transpose of the equation $AX + X^TB = C$, we have $B^TX + X^TA^T = C^T$. Note that $\sigma(A(B^T)^{-1}) = \sigma(B^TA^{-1})$. Moreover, $\lambda\mu \neq 1$ for any $\lambda, \mu \in \sigma(B^TA^{-1})$ is equivalent to $\lambda\mu \neq 1$ for any $\lambda, \mu \in \sigma(A(B^T)^{-1})$. Thus, we have the following corollaries. The proof of these corollaries is similar to the above theorems and is omitted here.

Corollary 10.1. Supposed that B is nonsingular, let

$$g_{(I_n, A(B^T)^{-1})}(s) := \det(I_n - sA(B^T)^{-1}) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0, \alpha_0 = 1, \tag{10.1.10}$$

and

$$\operatorname{adj}(I_n - sA(B^T)^{-1}) := \Phi_{n-1}s^{n-1} + \cdots + \Phi_1 s + \Phi_0. \tag{10.1.11}$$

(1) If the equation (10.0.1) has solution, then the solution can be stated as

$$Xg_{(I_n, A(B^T)^{-1})}(B^{-1}A^T) = (B^T)^{-1}\sum_{i=0}^{n-1}\Phi_i(C^T - CB^{-1}A^T)(B^{-1}A^T)^i. \tag{10.1.12}$$

(2) If $\lambda\mu \neq 1$ for any $\lambda, \mu \in \sigma(A(B^T)^{-1})$, then the unique solution to (10.0.1) can be characterized by

$$X = (B^T)^{-1}\Big[\sum_{i=0}^{n-1}\Phi_i(C^T - CB^{-1}A^T)(B^{-1}A^T)^i\Big]g^{-1}_{(I_n, A(B^T)^{-1})}(B^{-1}A^T). \quad (10.1.13)$$

Corollary 10.2. Supposed that B is nonsingular and $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of $A(B^T)^{-1}$.

(1) Let X be a solution to (10.0.1), then the solution can be summarized as

$$\begin{aligned} &Xg_{(I_n, A(B^T)^{-1})}(B^{-1}A^T) \\ &\quad = (B^T)^{-1}\sum_{j=0}^{n-1}\sum_{k=0}^{j}\beta_k(A(B^T)^{-1})^{j-k}(C^T - CB^{-1}A^T)(B^{-1}A^T)^j. \end{aligned} \quad (10.1.14)$$

(2) Let $\mu_i\mu_j \neq 1$ for all i, j, then the unique solution to (10.0.1) can be characterized with

$$\begin{aligned} X = (B^T)^{-1}&\Big[\sum_{j=0}^{n-1}\sum_{k=0}^{j}\beta_k(A(B^T)^{-1})^{j-k}(C^T - CB^{-1}A^T)(B^{-1}A^T)^j\Big] \\ &\times g^{-1}_{(I_n, A(B^T)^{-1})}(B^{-1}A^T). \end{aligned} \quad (10.1.15)$$

Corollary 10.3. Supposed that B is nonsingular and $C^T - CB^{-1}A^T = C_1C_2$, $C_1 \in \mathcal{R}^{n\times r}$, $C_2 \in \mathcal{R}^{r\times n}$, where $r = \operatorname{rank}(C^T - CB^{-1}A^T)$.

(1) Let X be a solution to (10.0.1), then the solution can be characterized with

$$\begin{aligned} &Xg_{(I_n, A(B^T)^{-1})}(B^{-1}A^T) \\ &\quad = (B^T)^{-1}Q_c(A(B^T)^{-1}, C_1, n)S_r(I, A(B^T)^{-1})Q_o(B^{-1}A^T, C_2, n). \end{aligned} \quad (10.1.16)$$

(2) Let $\lambda\mu \neq 1$ for any $\lambda, \mu \in \sigma(A(B^T)^{-1})$, then a unique solution to (10.0.1) is

$$X = (B^T)^{-1}Q_c(A(B^T)^{-1}, C_1, n)S_r(I, A(B^T)^{-1})Q_o(B^{-1}A^T, C_2, n)g^{-1}_{(I_n, A(B^T)^{-1})}(B^{-1}A^T). \quad (10.1.17)$$

10.1.2 The Second Case: A and B are Nonsingular

Theorem 10.4. Let A and B be nonsingular,

$$h_{B^TA^{-1}}(s) := \det(sI_n - B^TA^{-1}) = \gamma_n s^n + \cdots + \gamma_1 s + \gamma_0, \gamma_n = 1, \quad (10.1.18)$$

and

$$\mathrm{adj}(sI_n - B^TA^{-1}) := D_{n-1}s^{n-1} + \cdots + D_1 s + D_0. \tag{10.1.19}$$

(1) Supposed that X is a solution to the equation (10.0.1), then the solution can be characterized with

$$Xh_{B^TA^{-1}}(B^{-1}A^T) = A^{-1}\sum_{i=0}^{n-1} D_i(CB^{-1}A^T - C^T)(B^{-1}A^T)^i. \tag{10.1.20}$$

(2) If $\sigma(B^TA^{-1}) \cap \sigma(B^{-1}A^T) = \emptyset$, then the equation (10.0.1) has a unique solution

$$X = A^{-1}\Big[\sum_{i=0}^{n-1} D_i(CB^{-1}A^T - C^T)(B^{-1}A^T)^i\Big] \times [h_{B^TA^{-1}}(B^{-1}A^T)]^{-1}. \tag{10.1.21}$$

Proof. According to definition (10.1.18), the equation (10.0.1) can be rewritten as

$$\mathrm{Syl}(sI_n - B^TA^{-1}, B^{-1}A^T, AX) = CB^{-1}A^T - C^T.$$

By definitions (10.1.18) and (10.1.19), the above relation is equivalent to

$$\Phi[sI_n - B^TA^{-1}]\mathrm{vec}(AX) = \mathrm{vec}(CB^{-1}A^T - C^T).$$

It follows from lemma 10.2 that

$$\Phi[I_n\det(sI_n - B^TA^{-1})]\mathrm{vec}(AX) = \Phi[\mathrm{adj}(sI_n - B^TA^{-1})]\mathrm{vec}(CB^{-1}A^T - C^T).$$

So we get

$$\mathrm{Syl}(I_n\det(sI_n - B^TA^{-1}), B^{-1}A^T, AX) = \mathrm{Syl}(\mathrm{adj}(sI_n - B^TA^{-1}), B^{-1}A^T, CB^{-1}A^T - C^T).$$

From definition (10.1.18) and some simple computations, we have

$$\mathrm{Syl}(I_n\det(sI_n - B^TA^{-1}), B^{-1}A^T, AX) = AXh_{B^TA^{-1}}(B^{-1}A^T),$$

and

$$\mathrm{Syl}(\mathrm{adj}(sI_n - B^TA^{-1}), B^{-1}A^T, CB^{-1}A^T - C^T) = \sum_{i=0}^{n-1} D_i(CB^{-1}A^T - C^T)(B^{-1}A^T)^i.$$

Thus, we get

$$Xh_{B^TA^{-1}}(B^{-1}A^T) = A^{-1}\sum_{i=0}^{n-1} D_i(CB^{-1}A^T - C^T)(B^{-1}A^T)^i.$$

Therefore, (10.1.20) has been proved. Due to $\sigma(B^TA^{-1})\cap\sigma(B^{-1}A^T)=\varnothing$, $h_{B^TA^{-1}}(B^{-1}A^T)$ is nonsingular and the equation (10.0.1) has a unique solution. So (10.1.21) holds. □

Next, the well-known Faddeev-Leverrier algorithm[186] is shown as follows:

$$\begin{cases} D_{n-k}=D_{n-k+1}B^TA^{-1}+\gamma_{n-k+1}I_n, & D_n=0, k=1,2,\ldots,n,\\ \gamma_{n-k}=\dfrac{\text{trace}(D_{n-k}B^TA^{-1})}{k}, & \gamma_n=1, k=1,2,\ldots,n. \end{cases} \tag{10.1.22}$$

Based on this Faddeev-Leverrier algorithm, we have the equivalent forms of theorem 10.4. The proof of that is similar to theorems 10.2 and 10.3 and it is omitted here.

Theorem 10.5. Supposed that A and B are nonsingular.

(1) If X is a solution of the equation (10.0.1), then the solution can be stated as

$$Xh_{B^TA^{-1}}(B^{-1}A^T)=A^{-1}\sum_{k=1}^{n}\sum_{j=0}^{k-1}\gamma_k(B^TA^{-1})^j(CB^{-1}A^T-C^T)(B^{-1}A^T)^{k-j-1}. \tag{10.1.23}$$

(2) If $\sigma(B^TA^{-1})\cap\sigma(B^{-1}A^T)=\varnothing$, then the equation (10.0.1) has a unique solution

$$X=A^{-1}\Big[\sum_{k=1}^{n}\sum_{j=0}^{k-1}\gamma_k(B^TA^{-1})^j(CB^{-1}A^T-C^T)(B^{-1}A^T)^{k-j-1}\Big]h^{-1}_{B^TA^{-1}}(B^{-1}A^T). \tag{10.1.24}$$

Remark 3. Different from theorem 10.4, in order to obtain the solution to (10.0.1), this theorem only needs the coefficients of the characteristic polynomial of B^TA^{-1}. Besides the Faddeev-Leverrier algorithm, there exist other numerically reliable algorithms which can also be applied.

Theorem 10.6. Supposed that A and B are nonsingular, let $CB^{-1}A^T-C^T=\breve{C}_1\breve{C}_2$ hold, $\breve{C}_1\in\mathcal{R}^{n\times r}$, $\breve{C}_2\in\mathcal{R}^{r\times n}$, where $r=\text{rank}(CB^{-1}A^T-C^T)$.

(1) If X is a solution to the equation (10.0.1), then the solution can be expressed as

$$Xh_{B^TA^{-1}}(B^{-1}A^T)=A^{-1}Q_c(B^TA^{-1},\breve{C}_1,n)S_r(B^TA^{-1})Q_o(B^{-1}A^T,\breve{C}_2,n). \tag{10.1.25}$$

(2) If $\sigma(B^TA^{-1})\cap\sigma(B^{-1}A^T)=\varnothing$, then the equation (10.0.1) has a unique solution

$$X=A^{-1}Q_c(B^TA^{-1},\breve{C}_1,n)S_r(B^TA^{-1})Q_o(B^{-1}A^T,\breve{C}_2,n)h^{-1}_{B^TA^{-1}}(B^{-1}A^T). \tag{10.1.26}$$

10.2 Solutions to the Generalized Sylvester Transpose Matrix Equation

In this section, we discuss the parametric solutions to the equation (10.0.2). Firstly, we define

$$f_{(I_n,-A^{-1}B^T)}(s) := \det(I+sA^{-1}B^T) = \delta_n s^n + \cdots + \delta_1 s + \delta_0, \delta_0 = 1, \tag{10.2.1}$$

and

$$\operatorname{adj}(I_n + sA^{-1}B^T) := M_{n-1}s^{n-1} + \cdots + M_1 s + M_0. \tag{10.2.2}$$

Theorem 10.7. If A is nonsingular and $\{s|\det(I+sA^{-1}B^T)=0\} \bigcap \sigma(-(A^{-1})^T B) = \emptyset$ holds, then all the solutions to the equation (10.0.2) are stated as

$$\begin{cases} X = \sum\limits_{i=0}^{n-1} M_i A^{-1} CZ(-(A^{-1})^T B)^i \\ \qquad + \sum\limits_{i=0}^{n-1} A^{-1}(-B^T A^{-1})^i (M_i A^{-1} CZ)^T(-B), \\ Y = Zf_{(I_n,-A^{-1}B^T)}(-(A^{-1})^T B). \end{cases} \tag{10.2.3}$$

Proof. First, we'll prove that X and Y given in (10.2.3) are the solutions to (10.0.2). By calculation we get

$$\begin{aligned} & AX + X^T B \\ & = A\sum_{i=0}^{n-1} M_i A^{-1} CZ((A^{-1})^T(-B))^i + \sum_{i=0}^{n-1}(-B^T A^{-1})^i (M_i A^{-1} CZ)^T B \\ & \quad + A\sum_{i=0}^{n-1} A^{-1}(-B^T A^{-1})^i (M_i A^{-1} CZ)^T(-B) \\ & \quad + \sum_{i=0}^{n-1}(-B)^T(M_i A^{-1} CZ)(-(A^{-1})^T B)^i (A^{-1})^T B \\ & = \sum_{i=0}^{n-1} AM_i A^{-1} CZ(-(A^{-1})^T B)^i - \sum_{i=0}^{n-1}(-B^T)M_i A^{-1} CZ(-(A^{-1})^T B)^{i+1} \\ & = AM_0 A^{-1} CZ + \sum_{i=1}^{n-1}(AM_i + B^T R_{i-1})A^{-1} CZ(-(A^{-1})^T B)^i \\ & \quad + B^T M_{n-1} A^{-1} CZ(-(A^{-1})^T B)^n. \end{aligned}$$

Note that $(I_n + sA^{-1}B^T)\mathrm{adj}(I_n + sA^{-1}B^T) = I_n\det(I_n + sA^{-1}B^T)$, we have

$$\begin{cases} M_0 = \delta_0 I_n, \\ M_i + A^{-1}B^T R_{i-1} = \delta_i I_n, \quad i \in I[1, n-1], \\ A^{-1}B^T M_{n-1} = \delta_n I_n. \end{cases}$$

Thus, one get

$$\begin{aligned} &AM_0A^{-1}CZ + \sum_{i=1}^{n-1}(AM_i + B^TM_{i-1})A^{-1}CZ(-(A^{-1})^TB)^i + B^TM_{n-1}A^{-1}CZ(-(A^{-1})^TB)^n \\ &= CZ\sum_{i=0}^{n}\delta_i(-(A^{-1})^TB)^i = CZf_{(I_n,-A^{-1}B^T)}(-(A^{-1})^TB) = CY. \end{aligned}$$

Therefore, X and Y in (10.2.3) are satisfied to (10.0.2).

Next, we will prove the completeness of (10.2.3). Firstly, it is showed that (10.2.3) has rp parameters which are represented by the elements of the parametric matrix Z. Secondly, because $\{s|\det(I_n + sA^{-1}B^T) = 0\}\bigcap\sigma(-B^TA^{-1}) = \varnothing$ holds and $f_{(I_n,-A^{-1}B^T)}(-(A^{-1})^TB)$ is nonsingular, this demonstrates that the mapping $Z \to (X, Y)$ given in (10.2.3) is injective. So all the solutions to (10.0.2) can be characterized by (10.2.3). $\square$

In the following, we'll give two equivalent forms of (10.2.3), one of which is stated as the controllability matrix associated with $A^{-1}B^T$ and $A^{-1}C$, symmetric operator matrix, and the observability matrix associated with $(A^{-1})^TB$ and Z. Using this form, convenience and advantages may be provided for further analyzing control problem.

Theorem 10.8. If A is nonsingular and $\{s|\det(I_n + sA^{-1}B^T) = 0\}\bigcap\sigma(-(A^{-1})^TB) = \varnothing$ holds, then all the solutions to the equation (10.0.2) can be characterized by

$$\begin{cases} X = \displaystyle\sum_{i=0}^{n-1}\sum_{k=0}^{i}\delta_k(-A^{-1}B^T)^{i-k}A^{-1}CZ(-(A^{-1})^TB)^i + A^{-1} \\ \qquad \times \Big[\displaystyle\sum_{i=0}^{n-1}\sum_{k=0}^{i}\delta_k(-A^{-1}B^T)^{i-k}A^{-1}CZ(-(A^{-1})^TB)^i\Big]^T(-B), \\ Y = Zf_{(I_n,-A^{-1}B^T)}(-(A^{-1})^TB), \end{cases} \tag{10.2.4}$$

or

$$\begin{cases} X = Q_c(-A^{-1}B^T, A^{-1}C, n)S_r(I, -A^{-1}B^T)Q_o(-(A^{-1})^TB, Z, n) \\ \qquad + A^{-1}[Q_c(-A^{-1}B^T, A^{-1}C, n)S_r(I, -A^{-1}B^T) \\ \qquad \times Q_o(-(A^{-1})^TB, Z)]^T(-B), \\ Y = Zf_{(I_n,-A^{-1}B^T)}(-(A^{-1})^TB), \end{cases} \tag{10.2.5}$$

where $Z \in \mathcal{R}^{n\times n}$ is an arbitrary chosen parameter matrix.

Proof. From (10.1.5), one derive

$$\begin{cases} M_0 = I_n, \\ M_i = -A^{-1}B^TM_{i-1} + \alpha_i I_n, \quad i \in I[1, n-1], \\ \delta_i = \dfrac{\operatorname{trace}(-A^{-1}B^TM_{i-1})}{i}, \quad i \in I[1, n]. \end{cases}$$

This relation can be compactly expressed as

$$\begin{cases} M_0 = I_n, \\ M_1 = \delta_1 I_n + (-A^{-1}B^T), \\ M_2 = \delta_2 I_n + \delta_1(-A^{-1}B^T) + (-A^{-1}B^T)^2, \\ \cdots \\ M_{n-1} = \delta_{n-1}I_n + \delta_{n-2}(-A^{-1}B^T) + \cdots + (-A^{-1}B^T)^{n-1}. \end{cases}$$

Also it can be rewritten as

$$M_i = \sum_{k=0}^{i} \delta_k(-A^{-1}B^T)^{i-k}, \delta_0 = 1, i = 1, 2, \ldots, n-1.$$

Thus, by (10.2.3) we get

$$\sum_{i=0}^{n-1} M_iA^{-1}CZ(-(A^{-1})^TB)^i = \sum_{i=0}^{n-1}\sum_{k=0}^{i} \delta_k(-A^{-1}B^T)^{i-k}A^{-1}CZ(-(A^{-1})^TB)^i.$$

and

$$\sum_{i=0}^{n-1}(-B^TA^{-1})^i(M_iA^{-1}CZ)^T = \left[\sum_{i=0}^{n-1}\sum_{k=0}^{i} \delta_k(-A^{-1}B^T)^{i-k}A^{-1}CZ(-(A^{-1})^TB)^i\right]^T.$$

Moreover, from (10.2.3) we have

$$\begin{aligned} &\sum_{i=0}^{n-1} M_iA^{-1}CZ(-(A^{-1})^TB)^i \\ &\quad = Q_c(-A^{-1}B^T, A^{-1}C, n)S_r(I, -A^{-1}B^T)Q_o(-(A^{-1})^TB, Z, n), \end{aligned}$$

and

$$\begin{aligned} &A^{-1}\sum_{i=0}^{n-1}(-B^TA^{-1})^i(M_iA^{-1}CZ)^T(-B) \\ &\quad = A^{-1}[Q_c(-A^{-1}B^T, A^{-1}C, n)S_r(I, -A^{-1}B^T)Q_o(-(A^{-1})^TB, Z, n)]^T(-B). \end{aligned}$$

Therefore, the desired result follows. □

Suppose that B is nonsingular, we have the following results. Due to the line of the proof is similar to theorems 10.7 and 10.8, here it is omitted.

Theorem 10.9. Supposed that B is nonsingular and $\{s | \det(I + sA\ (B^{-1})^T) = 0\} \bigcap \sigma(-A^T\ B^{-1}) = \varnothing$ holds, let

$$f_{(I_n,-A(B^{-1})^T)}(s) := \det(I_n + sA(B^{-1})^T) = \omega_n s^n + \cdots + \omega_1 s + \omega_0, \alpha_0 = 1, \tag{10.2.6}$$

and

$$\operatorname{adj}(I_n + sA(B^{-1})^T) := N_{n-1}s^{n-1} + \cdots + N_1 s + N_0, \tag{10.2.7}$$

then all the solutions to the equation (10.0.2) can be expressed as

$$\begin{cases} X = \sum\limits_{i=0}^{n-1} \left[N_i CZ(-A^T B^{-1})^i \right]^T + \sum\limits_{i=0}^{n-1} (B^{-1})^T (N_i CZ)(-A^T B^{-1})^i (-A)^T, \\ Y = Zf_{(I_n,-A(B^{-1})^T)}(-A^T B^{-1})B. \end{cases} \tag{10.2.8}$$

where $Z \in \mathcal{R}^{n\times n}$ is an arbitrary chosen parameter matrix.

Theorem 10.10. Supposed that B is nonsingular and $\{s | \det(I + sA(B^{-1})^T) = 0\} \bigcap \sigma(-A^T\ B^{-1}) = \varnothing$ holds, then all the solutions to the equation (10.0.2) can be stated as

$$\begin{cases} X = \sum\limits_{i=0}^{n-1} \sum\limits_{k=0}^{i} \left[\omega_k (-A(B^{-1})^T)^{i-k} CZ(-A^T B^{-1})^i \right]^T \\ \qquad + \sum\limits_{i=0}^{n-1} \sum\limits_{k=0}^{i} \omega_k (B^{-1})^T (-A(B^{-1})^T)^{i-k} CZ(-A^T B^{-1})^i (-A)^T, \\ Y = Zf_{(I_n,-A(B^{-1})^T)}(-A^T B^{-1})B, \end{cases} \tag{10.2.9}$$

or

$$\begin{cases} X = \left[Q_c(-A(B^{-1})^T, C, n) S_r(I, -A(B^{-1})^T) Q_o(-A^T B^{-1}, Z) \right]^T \\ \qquad + (B^{-1})^T \Big[Q_c(-A(B^{-1})^T, C, n) S_r(I, -A(B^{-1})^T) \\ \qquad Q_o(-A^T B^{-1}, Z, n) \Big] (-A)^T, \\ Y = Zf_{(I_n,-A(B^{-1})^T)}(-A^T B^{-1})B, \end{cases} \tag{10.2.10}$$

where $Z \in \mathcal{R}^{n\times n}$ is an arbitrary chosen parameter matrix.

10.3 Algorithms for Solving Two Transpose Equations and Numerical Example

Through the discussion of sections 10.1 and 10.2, it is shown that the solutions to (10.0.1) and (10.0.2) can be constructed according to the presented new results. In this section, three algorithms are proposed for computing the solutions to (10.0.1) and (10.0.2).

Algorithm 10.1. *(For equation (10.0.1). Case 1: A is nonsingular.)*

- *Step 1. Find the decomposition of the matrix $C-(B^TA^{-1}C)^T$, obtain C_1 and C_2, then compute $Q_c(B^TA^{-1},C_1,n)$ and $Q_o((B^TA^{-1})^T,C_2,n)$.*
- *Step 2. Compute A^{-1},B^TA^{-1} and $f_{B^TA^{-1}}(s)$, find $\alpha_1,\alpha_2,\ldots,\alpha_n$, then calculate $S_r(I,B^TA^{-1})$ and $f_{(I_n,B^TA^{-1})}((B^TA^{-1})^T)$.*
- *Step 3. Find the solution to the equation (10.0.1) by the formula (10.1.8); For any $\lambda,\mu\in\sigma(B^TA^{-1})$, if $\lambda\mu\neq 1$, then compute $f^{-1}_{(I_n,B^TA^{-1})}((B^TA^{-1})^T)$, obtain the unique solution of the equation (10.0.1) by the formula (10.1.9).*

Algorithm 10.2. *(For equation (10.0.1). Case 2: A and B are nonsingular.)*

- *Step 1. Find the decomposition of the matrix $CB^{-1}A^T-C^T$, obtain $\breve{C}_1$ and $\breve{C}_2$, then compute $Q_c(B^TA^{-1},\breve{C}_1,n)$ and $Q_o((B^TA^{-1})^T,\breve{C}_2,n)$.*
- *Step 2. Compute B^{-1},B^TA^{-1} and $h_{B^TA^{-1}}(s)$, find $\gamma_1,\gamma_2,\ldots,\gamma_n$, then calculate $g_{B^TA^{-1}}(B^{-1}A^T)$ and $S_r(B^TA^{-1})$.*
- *Step 3.Find the solution to the equation (10.0.1) by the formula (10.1.24); If $\sigma(B^TA^{-1})\cap\sigma(B^{-1}A^T)=\varnothing$, compute $h^{-1}_{B^TA^{-1}}(B^{-1}A^T)$, then calculate the unique solution of the equation (10.0.1) by the formula (10.1.26).*

Remark 4. If $f_{B^TA^{-1}}(s):=\det(sI_n-B^TA^{-1})^{-1})=\sum_{i=0}^n\alpha_is^i$, then $f_{(I_n,B^TA^{-1})}(s):=\det(I_n-sB^TA^{-1})=\sum_{i=0}^n\alpha_is^{n-i}$. Because the order of the coefficients in $f_{B^TA^{-1}}(s)$, *i.e.* $\alpha_0,\alpha_1,\ldots,\alpha_n$, is opposite to that of $f_{(I_n,B^TA^{-1})}(s)$.

Remark 5. If B is nonsingular, it has an algorithm similar to algorithm 10.1 and here is omitted. If A is nonsingular, we can obtain the solution to (10.0.1) by algorithm 10.1 or algorithm 10.2. Similarly, if B is nonsingular, the solution to (10.0.1) can also be obtained by two algorithms.

Example 10.1. We consider the explicit solution to the Sylvester transpose matrix equation in the form of (10.0.1) with the following parameters:

$$A=\begin{pmatrix}3&11&-1\\2&10&8\\1&0&1\end{pmatrix},\quad B=\begin{pmatrix}-1&4&0\\1&-3&6\\2&3&-7\end{pmatrix},\quad C=\begin{pmatrix}-19&19&24\\16&78&12\\13&-22&33\end{pmatrix}.$$

The first method: by algorithm 10.1

By simple computation, the eigenvalues of the matrix B^TA^{-1} are as follows:

$$\lambda_1=0.1468,\quad \lambda_2=-1.2951+1.7362i,\quad \lambda_3=-1.2951-1.7362i.$$

It is easy to check that $\lambda_i\lambda_j\neq 1,i,j=1,2,3$. So the equation (10.0.1) has a unique solution. From algorithm 10.1, by some computations, we have

$$Q_o((B^TA^{-1})^T, C_2, 3) = 10^3 \times \begin{pmatrix} 0.0616 & -0.1953 & -0.0275 \\ -0.0772 & 0.1558 & 0.1045 \\ 0.0313 & -0.0282 & -0.0553 \\ -0.0541 & -0.0904 & 0.2142 \\ -0.0346 & 0.4590 & -0.3641 \\ 0.0394 & -0.2537 & 0.1559 \\ -0.2376 & 1.0221 & -0.4373 \\ 0.5390 & -1.7938 & 0.4641 \\ -0.2563 & 0.7784 & -0.1455 \end{pmatrix},$$

$$f_{(I_3, B^TA^{-1})}(s) = -0.6887s^3 + 4.3113s^2 + 2.4434s + 1,$$

$$C - (B^TA^{-1}C)^T = C_1C_2, \, C_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 4 & 5 & 6 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 72.2812 & -107.7931 & 48.0212 \\ -36.6366 & -24.4032 & 3.8939 \\ 38.8859 & 25.5013 & -62.1857 \end{pmatrix},$$

$$S_3(I, B^TA^{-1}) = \begin{pmatrix} I_3 & 2.4434I_3 & 4.3113I_3 \\ 0_3 & I_3 & 2.4434I_3 \\ 0_3 & 0_3 & I_3 \end{pmatrix},$$

and

$$\begin{aligned} &Q_c(B^TA^{-1}, C_1, 3) \\ &= \left(\begin{matrix} 1.0000 & 1.0000 & 0 & -4.3019 & -4.6226 \\ 0 & 2.0000 & 4.0000 & 18.5660 & 22.7925 \\ 4.0000 & 5.0000 & 6.0000 & -7.0000 & -10.0000 \end{matrix} \right. \\ &\qquad \left. \begin{matrix} -4.7170 & 14.5867 & 19.1912 & 24.7896 \\ 27.0943 & -36.3265 & -51.2004 & -70.7682 \\ -14.0000 & 0.4151 & 3.9811 & 9.7358 \end{matrix} \right). \end{aligned}$$

Thus, by theorem 10.3, the unique solution to the equation (10.0.1) is

$$X = \begin{pmatrix} 0.9999 & 2.0007 & -3.0005 \\ -2.0000 & -0.0001 & 6.0001 \\ 3.0000 & 5.9999 & 0.0001 \end{pmatrix}.$$

The second method: by algorithm 10.2

Similarly, the eigenvalues of the matrix B^TA^{-1} can be obtained by simple computation:

$$\gamma_1 = 0.1468, \gamma_2 = -1.2951 + 1.7362i, \gamma_3 = -1.2951 - 1.7362i.$$

Due to $\gamma_i\gamma_j \neq 1, i, j = 1, 2, 3,$ equation (10.0.1) has a unique solution. By algorithm 10.2, by some computations one gets

$$CB^{-1}A^T - C^T = \breve{C}_1\breve{C}_2,$$

$$g_{B^TA^{-1}}(s) = s^3 + 2.4434s^2 + 4.3113s - 0.6887,$$

$$Q_o(B^{-1}A^T, \breve{C}_2, 3) = \begin{pmatrix} 27.8192 & -94.7192 & 6.1628 \\ -9.3914 & 46.5595 & 2.4884 \\ -8.2260 & -12.6490 & 21.9535 \\ -41.8461 & -10.1242 & -21.4734 \\ 49.0233 & 42.0930 & 12.7442 \\ -10.3864 & -44.8060 & -4.9685 \\ -229.5391 & -330.9403 & -21.2978 \\ 296.5836 & 431.9032 & 33.9439 \\ -130.3930 & -171.9269 & -19.7306 \end{pmatrix},$$

$$\breve{C}_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \\ 4 & 5 & 2 \end{pmatrix}, \breve{C}_2 = \begin{pmatrix} 27.8192 & -94.7192 & 6.1628 \\ -9.3914 & 46.5595 & 2.4884 \\ -8.2260 & -12.6490 & 21.9535 \end{pmatrix},$$

$$S_3(B^TA^{-1}) = \begin{pmatrix} 4.3113I_3 & 2.4434I_3 & I_3 \\ 2.4434I_3 & I_3 & 0_3 \\ I_3 & 0_3 & 0_3 \end{pmatrix},$$

and

$$\begin{aligned} &Q_c(B^TA^{-1}, \breve{C}_1, 3) \\ &= \left(\begin{matrix} 1.0000 & 2.0000 & 3.0000 & -4.3019 & -4.1509 \\ 0 & 4.0000 & -1.0000 & 18.5660 & 22.2830 \\ 4.0000 & 5.0000 & 2.0000 & -7.0000 & -10.0000 \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} -3.0660 & 14.5867 & 18.9066 & 4.4538 \\ 9.3113 & -36.3265 & -51.1255 & -4.1192 \\ -0.5000 & 0.4151 & 4.7075 & -6.7217 \end{matrix}\right). \end{aligned}$$

Thus, by theorem 10.6, the unique solution to equation (10.0.1) is

$$X = \begin{pmatrix} 0.9999 & 2.0001 & -3.0004 \\ -2.0000 & -0.0001 & 6.0000 \\ 2.9999 & 5.9999 & 0.0000 \end{pmatrix}.$$

Algorithm 10.3. *(For equation (10.0.2). Case: A is nonsingular.)*

- *Step 1. Compute* $A^{-1}, A^{-1}B^T$ *and* $A^{-1}C$, *then calculate* $Q_c(-A^{-1}B^T, A^{-1}C, n)$.
- *Step 2. Compute* $f_{(-A^{-1}B^T)}(s)$, *find* $\delta_1, \delta_2, \ldots, \delta_n$, *then calculate* $f_{(I_n, -A^{-1}B^T)}((A^{-1})^T B)$ *and* $S_r(I, -A^{-1}B^T)$.
- *Step 3. Choose a parameter matrix* Z, *then compute* $Q_o(-(A^{-1})^T B, Z, n)$.
- *Step 4. Find a solution to the equation (10.0.2) by the formula (10.2.5).*

Remark 6. If B is nonsingular, it has a similar algorithm to algorithm 10.3 and here is omitted.

Example 10.2. In this example, we consider the parametric solutions to equation (10.0.2) with the following parametric matrices:

$$A = \begin{pmatrix} 4 & 11 & -2 & 10 \\ 21 & 0 & 80 & 2 \\ 1 & 10 & 11 & 2 \\ 2 & 3 & 21 & 4 \end{pmatrix}, C = \begin{pmatrix} -1 & 9 & 4 & 2 \\ 6 & 7 & 12 & 1 \\ 3 & -2 & 3 & 12 \\ 2 & 3 & 23 & 12 \end{pmatrix},$$

$$B = \begin{pmatrix} -11 & 42 & 10 & 2 \\ 12 & -23 & 6 & 3 \\ 21 & 30 & -17 & 2 \\ 23 & 1 & 0 & 2 \end{pmatrix}.$$

By some computations, we get

$$f_{(I_4, B^T A^{-1})}(s) = 1.8048s^4 + 6.7717s^3 + 6.0306s^2 + 2.8223s + 1.$$

Choose Z as

$$Z = \begin{pmatrix} 111.2117 & 794.7075 & 233.5488 & 188.2771 \\ 13.8692 & 149.9002 & 40.5172 & 33.7128 \\ 2.1329 & 5.5192 & 2.4184 & 1.6934 \\ -1.6508 & -10.8792 & -3.9105 & -2.6594 \end{pmatrix}.$$

Thus, we get

$$
\begin{aligned}
&Q_c(-A^{-1}B^T, A^{-1}C, 4)\\
&= \left(\begin{matrix}
-0.1776 & 0.4605 & -3.4495 & -2.2801 & 0.3487 & -2.1196 & -4.7172 & 0.4709\\
0.2278 & -0.4081 & -1.0575 & 0.6707 & 0.3064 & -0.2802 & 6.4886 & 3.3105\\
0.1280 & -0.0622 & 0.9770 & 0.5987 & 0.0380 & 0.2143 & 2.7520 & 1.2676\\
-0.2539 & 1.1523 & 3.1384 & 0.4940 & -0.6225 & -0.3245 & -16.4958 & -9.2827
\end{matrix}\right.
\end{aligned}
$$

$$
\left.\begin{matrix}
0.6664 & 1.4451 & 23.7628 & 10.9121 & -2.2126 & 5.9847 & -11.4997 & -15.9334\\
-0.5057 & 2.7668 & 5.6111 & -0.4947 & -0.9304 & -1.9951 & -33.2165 & -15.8537\\
-0.3162 & 0.7060 & -2.7953 & -2.7906 & 0.1772 & -1.9370 & -8.2816 & -1.2043\\
1.5943 & -5.1792 & 2.9494 & 10.8548 & 0.2810 & 8.1124 & 57.9738 & 17.0626
\end{matrix}\right),
$$

and

$$
Q_o(-(A^{-1})^TB, Z, 4) = \begin{pmatrix}
111.2117 & 794.7075 & 233.5488 & 188.2771\\
13.8692 & 149.9002 & 40.5172 & 33.7128\\
2.1329 & 5.5192 & 2.4184 & 1.6934\\
-1.6508 & -10.8792 & -3.9105 & -2.6594\\
-23.4868 & -283.6510 & -73.5257 & -65.3330\\
5.4046 & -57.8579 & -10.6996 & -10.6265\\
-2.1652 & -0.5315 & -1.0392 & -0.7826\\
0.1456 & 1.3795 & 1.1643 & 0.9756\\
0.5917 & 106.0737 & 11.5955 & 18.9204\\
-15.4207 & 25.8637 & -1.5306 & 1.5146\\
3.7945 & -1.1908 & 0.5386 & 0.5077\\
-3.0130 & 5.3683 & 1.5835 & -0.0638\\
19.2898 & -101.3619 & -6.4952 & -4.6183\\
24.7008 & -18.7178 & 6.5999 & 2.4232\\
-5.2777 & -1.8355 & -1.8841 & -0.7852\\
9.0960 & 0.1991 & -0.6903 & 0.2245
\end{pmatrix}.
$$

Thus, by theorem 10.6, the parametric solutions to equation (10.0.2) are

$$
X = \begin{pmatrix}
1 & 20 & 3 & 2\\
3 & 5 & 6 & 8\\
0 & 2 & 6 & 7\\
8 & 9 & 3 & -2
\end{pmatrix},
$$

and

$$
Y = \begin{pmatrix}
-0.7857 & 187.2857 & 96.7143 & 89.5000\\
34.1853 & 48.5108 & 27.6114 & 20.4951\\
-11.7356 & -4.7634 & -1.4671 & -0.2488\\
32.2445 & -14.1288 & -10.9599 & -2.4803
\end{pmatrix}.
$$

10.4 Application in Control Theory

As an application of the proposed solution to (10.0.1), in this section, we consider the continuous zeroing dynamics (CZD) design of the time-varying linear system. The design formula expressed in matrix form is defined as

$$\dot{L}(t) = -\lambda JMP \circ (L(t)), \tag{10.4.1}$$

and $JMP(\cdot) : R^{m\times n} \to R^{m\times n}$ is denoted as an array of *jmp* functions:

$$JMP(L(t)) = \begin{pmatrix} jmp(l_{11}(t)) & jmp(l_{12}(t)) & \cdots & jmp(l_{1n}(t)) \\ jmp(l_{21}(t)) & jmp(l_{22}(t)) & \cdots & jmp(l_{2n}(t)) \\ \vdots & \vdots & \ddots & \vdots \\ jmp(l_{n1}(t)) & jmp(l_{n2}(t)) & \cdots & jmp(l_{nn}(t)) \end{pmatrix}, \tag{10.4.2}$$

where $jmp(l_{ij}(t)) = \begin{cases} 1, & l_{ij}(t) > 0 \\ 0, & l_{ij}(t) \le 0 \end{cases}$. And the time derivative is

$$L(t) = \dot{A}(t)X(t) + A(t)\dot{X}(t) + \dot{X}(t)^T B(t) + X(t)^T \dot{B}(t). \tag{10.4.3}$$

Substituting (10.4.1) into (10.4.3) and arranging, one has the following time-varying linear system

$$A(t)\dot{X}(t) + \dot{X}(t)^T B(t) = C(t), \tag{10.4.4}$$

where $A(t), B(t), C(t) \in \mathcal{R}^{n\times n}$ are smoothly time-varying matrices, $X(t) \in \mathcal{R}^{n\times n}$ is the unknown time-varying matrix to be determined. Besides, $t \in [0, t_f] \subseteq [0, +\infty)$, and t_f denotes the final time. Thus, the equation (10.4.4) is equivalent to the equation $AX + X^T B = C$. We choose the following parametric matrices:

$$A = \mathrm{triu}(\mathrm{rand}(10,10),1) + \mathrm{diag}(1 + \mathrm{diag}(\mathrm{rand}(10))), \quad B = \mathrm{tril}(\mathrm{rand}(10,10),1),$$

$$C = \left(\begin{matrix} 12.0308 & 7.4456 & 2.1117 & 1.9960 & 7.4386 \\ 3.3343 & 13.8946 & 5.5906 & 2.7438 & 7.0838 \\ 5.7489 & 3.8058 & 8.8902 & 3.5619 & 1.1289 \\ 1.9270 & 4.5679 & 4.6981 & 15.0807 & 6.2074 \\ 3.4261 & 4.5089 & 4.4524 & 5.6138 & 8.3613 \\ 2.3090 & 7.7085 & 6.0933 & 4.2581 & 4.6963 \\ 6.9273 & 3.3946 & 7.0041 & 3.2479 & 2.0214 \\ 6.1090 & 3.4464 & 8.7568 & 5.7667 & 3.7945 \\ 4.8700 & 3.0501 & 3.8442 & 6.3470 & 5.2390 \\ 6.8190 & 3.2828 & 4.2765 & 5.4472 & 3.3497 \end{matrix} \right.$$

$$\left.\begin{matrix} 3.4554 & 2.4897 & 5.3519 & 4.1122 & 7.8176 \\ 6.5278 & 5.5624 & 5.6862 & 4.1144 & 8.9310 \\ 5.0187 & 5.8117 & 3.1962 & 4.9426 & 7.1911 \\ 4.7645 & 5.5780 & 5.5443 & 4.5838 & 4.7910 \\ 7.7872 & 1.9622 & 3.9063 & 6.6722 & 5.1343 \\ 11.0492 & 5.5616 & 2.6473 & 7.4494 & 1.7236 \\ 2.1400 & 12.2184 & 9.0972 & 5.3937 & 3.9903 \\ 2.8991 & 5.5384 & 10.8941 & 7.6649 & 2.5836 \\ 2.3227 & 3.3837 & 5.7356 & 14.7296 & 1.9510 \\ 1.9376 & 2.6750 & 6.7963 & 2.3251 & 10.0195 \end{matrix}\right).$$

It is easy to check that $\eta\gamma \neq 1$ for any $\eta, \gamma \in \sigma(B^T A^{-1})$ and A is nonsingular. Therefore, the equation $AX + X^T B = C$ has a unique solution. It follows from algorithm 10.1 that the unique solution to $AX + X^T B = C$ is

$$X = \begin{pmatrix} 6.0000 & 0.1544 & 0.5975 & 0.1249 & 0.7593 & 0.1636 & 0.8092 & 0.0205 & 0.7519 & 0.3174 \\ 0.0000 & 6.0000 & 0.3353 & 0.0244 & 0.7406 & 0.6660 & 0.7486 & 0.9237 & 0.2287 & 0.8145 \\ 0.0000 & 0.0000 & 6.0000 & 0.2902 & 0.7437 & 0.8944 & 0.1202 & 0.6537 & 0.0642 & 0.7891 \\ 0.0000 & 0.0000 & 0.0000 & 6.0000 & 0.1059 & 0.5166 & 0.5250 & 0.9326 & 0.7673 & 0.8523 \\ 0.0000 & 0.0000 & 0.0000 & 0.0001 & 6.0000 & 0.7027 & 0.3258 & 0.1635 & 0.6712 & 0.5056 \\ 0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & 6.0000 & 0.5464 & 0.9211 & 0.7152 & 0.6357 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0000 & 6.0000 & 0.7947 & 0.6421 & 0.9509 \\ 0.0000 & 0.0000 & 0.0000 & -0.0001 & 0.0000 & 0.0000 & 0.0000 & 6.0000 & 0.4190 & 0.4440 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 6.0000 & 0.0600 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 6.0000 \end{pmatrix}.$$

10.5 Conclusions

In this chapter, we have studied the explicit solutions to the equations $AX + X^T B = C$ and $AX + X^T B = CY$, which are named the Sylvester transpose matrix equation and the generalized Sylvester transpose matrix equation. The work of this chapter has the following contributions.

- Three forms of solutions to the Sylvester transpose matrix equation are provided. One of the solutions is the well-known Jameson's Theorem. Two algorithms are proposed for computing the solutions to this equation.
- Compared with the necessary and sufficient conditions for the existence of solutions to the Sylvester transpose matrix equation in[163], the condition in this chapter only needs to check the eigenvalues of $B^T A^{-1}$. Moreover, there is no limit to all the coefficient matrices, while in[163] it requests a coefficient matrix $C = C^T$.
- As an application of the proposed algorithm for the equation $AX + X^T B = C$, the continuous zeroing dynamics (CZD) design of a time-varying linear system is given.

- Parametric solutions to the generalized Sylvester transpose matrix equation have been constructed. An algorithm for obtaining the parametric solutions has been proposed.
- The idea in this chapter is also applicable to some other matrix equations with similar structures, *e.g.*, the Stein transpose matrix equation $X + AX^T B = C$ and generalized Stein transpose matrix equation $X + AX^T B = CY$. What we need is to make some small changes of specific coefficients with respect to X^T.

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